

Hierarchies of invariant spin models

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Abstract

In this paper we present classes of state sum models based on the recoupling theory of angular momenta of $SU(2)$ (and of its q -counterpart $U_q(sl(2))$, q a root of unity). Such classes are arranged in hierarchies depending on the dimension d , and include all known closed models, *i.e.* the Ponzano–Regge state sum and the Turaev–Viro invariant in dimension $d = 3$, the Crane–Yetter invariant in $d = 4$. In general, the recoupling coefficient associated with a d -simplex turns out to be a $\{3(d-2)(d+1)/2\}j$ symbol, or its q -analog.

Each of the state sums can be further extended to compact triangulations $(T^d, \partial T^d)$ of a PL -pair $(M^d, \partial M^d)$, where the triangulation of the boundary manifold is not kept fixed. In both cases we find out the algebraic identities which translate complete sets of topological moves, thus showing that all state sums are actually independent of the particular triangulation chosen. Then, owing to Pachner’s theorems, it turns out that classes of PL -invariant models can be defined in any dimension d .

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1 Introduction

In what follows we shall always consider either *closed* d -dimensional simplicial PL -manifolds or *compact* d -dimensional simplicial PL -pairs $(M^d, \partial M^d)$, where the triangulation on the boundary $(d-1)$ -manifold is not kept fixed.

Recall that a closed PL -manifold of dimension d is a polyhedron $M^d \cong |T^d|$, each point of which has a neighborhood, in M^d , PL -homeomorphic to an open set in \mathbb{R}^d . The symbol \cong denotes homeomorphism, T^d is the underlying (finite) simplicial complex and $|T^d|$ denotes the associated topological space. PL -manifolds are realized by simplicial manifolds under the equivalence relation generated by PL -homeomorphisms. In particular, two d -dimensional closed PL -manifolds $M_1^d \cong |T_1^d|$ and $M_2^d \cong |T_2^d|$ are PL -homeomorphic, or $M_1^d \cong_{PL} M_2^d$, if there exists a map $g : M_1^d \rightarrow M_2^d$ which is both a homeomorphism and a simplicial isomorphism. We shall use the notation

$$T^d \longrightarrow M^d \cong |T^d| \quad (1)$$

to denote a *particular triangulation* of the closed d -dimensional PL -manifold M^d .

A PL -invariant $Z[M^d]$ is a quantity which is independent of the particular triangulation chosen in (1). The value of such an invariant depends just on the PL -class of the closed manifold, namely it is the same for PL -homeomorphic manifolds. The previous definitions can be naturally extended to the case of a PL -pair $(M^d, \partial M^d)$ of dimension d according to:

$$(T^d, \partial T^d) \longrightarrow (M^d, \partial M^d) \cong (|T^d|, |\partial T^d|), \quad (2)$$

where ∂T^d is the unique triangulation induced on the $(d-1)$ -dimensional boundary PL -manifold ∂M^d by the chosen triangulation T^d in M^d . A PL -invariant $Z[(M^d, \partial M^d)]$ is a quantity which is independent of the particular triangulation chosen in (2). The value of such an invariant depends upon the PL -class of the pair, namely it is the same for PL -homeomorphic pairs (the reader may refer *e.g.* to [16] for more details on PL -topology).

The general setting of the content of this paper can be summarized as follows:

- *step 1*) Given a (suitable defined) PL -invariant state sum $Z[M^{d-1}]$ for a closed $(d-1)$ -dimensional PL -manifold M^{d-1} , we extend it to a state sum for a pair $(T^d, \partial T^d \equiv T^{d-1})$. This is achieved by assembling in a suitable way the *squared roots* of the symbols associated with the fundamental blocks in $Z[M^{d-1}]$ in order to pick up the recoupling symbol to be associated with the d -dimensional simplex; the dimension of the $SU(2)$ -labelled (or the q -colored) $(d-2)$ -simplices is kept fixed when passing from T^{d-1} to $T^d \supset T^{d-1}$.
- *step 2*) The state sum for $(T^d, \partial T^d)$ gives rise to a PL -invariant $Z[(M^d, \partial M^d)]$ owing to the fact that we can exploit a set of topological moves, the *elementary*

shellings of Pachner [10] (the algebraic identities associated with such moves in $d = 3$ were established in [2] and [3]).

- *step 3*) From the expression of $Z[(M^d, \partial M^d)]$ we can now extract a state sum for a *closed* triangulation T^d . The proof of its PL -invariance relies now on the algebrization in any dimension d of the *bistellar moves* introduced in [9]. The procedure turns out to be consistent with known results in dimension 3 (see [12], [18] and [4]) and in dimension 4 (see [8] and [5]), and provides us with a PL -invariant $Z[M^d]$ where each d -simplex in T^d is represented by a $\{3(d-2)(d+1)/2\}_j$ recoupling coefficient of $SU(2)$ (or by the corresponding q -analog).

The scheme we have outlined above gives us an algorithmic procedure for generating (different kinds of) invariants for closed manifolds in contiguous dimensions, namely $Z[M^{d-1}] \rightarrow Z[M^d]$. Furthermore, the (multi)–hierarchical structure underlying these classes of invariants is sketched below as an array:

$$dimension : \quad 2 \quad 3 \quad 4 \quad \dots \quad d \quad d+1$$

$$Z_{\chi}^2 \quad Z_{PR}^3$$

$$Z_{\chi}^3 \quad Z_{CY}^4$$

$$Z^4_\chi \quad \cdot \cdot \cdot$$

$$\begin{array}{ccc} \cdot & \cdot & \\ \cdot & \cdot & \\ & \cdot & \end{array} \quad \begin{array}{cc} Z^d & Z^{d+1} \end{array}$$

$$Z_{\chi}^{d+1}$$

The quantities $Z_\chi^d \equiv Z_\chi[M^d]$ on the first diagonal of the array, which we referred to in *step 1*), are invariants depending upon the Euler characteristic of the closed manifold M^d ; they can be defined both in the classical case of $SU(2)$ and in the case $q \neq 1$, (in this last case the notation $Z_\chi^d(q)$ should be more suitable). They are obtained in any dimension d by labelling the $(d-1)$ -simplices of the triangulations with the ranks of $SU(2)$ representations, $j = 0, 1/2, 1, 3/2, \dots$ (see *Section 4* for the general definition in the q -case). Notice that these invariants are in fact trivial in dimension $d = 2n + 1$ since here we are dealing exclusively with manifolds.

The hierarchy on the second diagonal of the array includes classical PL -invariants $Z^d \equiv Z[M^d]$ involving products of $\{3(d-2)(d+1)/2\}j$ symbols of $SU(2)$ as we said in *step 3*). In this case the labelling j have to be assigned to the $(d-2)$ -simplices of each triangulation (namely edges in $d=3$, triangles in $d=4$, and so on). Thus we recover the Ponzano–Regge model Z_{PR}^3 and the Ooguri–Crane–Yetter invariant $Z_{CY}^4(q=1)$; the other invariants are new. A similar remark holds true

also in the q -deformed context, where the hierarchy would be rewritten in terms of the counterparts $Z^d(q) \equiv Z[M^d](q)$ (and in particular we found the Turaev–Viro invariant in $d = 4$, see [18]).

Coming back to the relations between invariants lying in the same row of the array, they can be further analyzed in view of the extension of each $Z[M^d]$ to $Z[(M^d, \partial M^d)]$. Thus the first row can be read as a *PR*-model with Z_χ^2 on its boundary, the second row as a *CY*-model with Z_χ^3 on its boundary, while the other rows display new invariants for *PL*-pairs in each dimension.

As a consequence of the above remarks, the whole table (together with a similar q -table) can be reconstructed row by row just from the explicit form of the invariant Z_χ^d . Then, in a sense, it is not surprising that the invariant Z_{CY}^4 , having on its boundary $Z_\chi^3 = \text{const.}$ for any choice of ∂M^4 , turns out to be simply the discretized version of a combination of signature and Euler characteristic of M^4 (see [15]). On the other hand, the invariants Z^{2n+1} generated by non-trivial Z_χ^{2n} are expected to be related to suitable types of torsions, as happens in the 3-dimensional case (see *e.g.* [17]). However, the proper way to investigate the nature of the invariants in $d = 2n > 4$ and $d = 2n+1 > 3$ is by no doubts the search for explicit correspondences with some *TQFTs*. It turns out that the continuous counterparts of the classical Z^d are indeed *BF* theories, although in the even case the identification of the resulting invariant(s) does not seem so straightforward. Similar considerations apply to the $Z^d(q)$, which should be discretized versions of *BF* theories with suitable cosmological terms. These last issues will be discussed elsewhere.

For the sake of clarity in the exposition, the presentation does not follow exactly the schedule given in *steps 1–3* at the beginning, mainly owing to the fact that calculations in dimension $d > 4$ can be performed only by diagrammatical methods. Thus, as an illustration of the analytical approach, we present in *Section 2* a short review of [2] and [3], while in *Section 3* we provide the extension of the *CY*-invariant ($q = 1$) to the case of a pair $(M^4, \partial M^4)$ and the expression of the induced Z_χ^3 on the boundary. In the following *Section 4* we give the explicit form of Z_χ^d for any closed M^d . In *Section 5* we generate $Z[(M^d, \partial M^d)]$ from Z_χ^{d-1} and we show its *PL*-invariance. *Section 6* contains the proof that the state sum induced by $Z[(M^d, \partial M^d)]$ when $\partial M^d = \emptyset$ is a well defined *PL*-invariant. Finally, in *Appendix B* we collect the basic notations in view of the extension to the $Z^d(q)$ hierarchy.

2 Ponzano–Regge model for $(M^3, \partial M^3)$ and induced 2-dimensional invariant

Following [2] and [3], the connection between a recoupling scheme of $SU(2)$ angular momenta and the combinatorial structure of a compact, 3-dimensional simplicial pair $(M^3, \partial M^3)$ can be established by considering *colored* triangulations which allow us to specialize the map (2) according to

$$(T^3(j), \partial T^3(j', m)) \longrightarrow (M^3, \partial M^3). \quad (3)$$

This map represents a triangulation associated with an admissible assignment of both spin variables to the collection of the edges $((d-2)$ -simplices) in $(T^3, \partial T^3)$ and of momentum projections to the subset of edges lying in ∂T^3 . The collective variable $j \equiv \{j_A\}$, $A = 1, 2, \dots, N_1$, denotes all the spin variables, n'_1 of which are associated with the edges in the boundary (for each A : $j_A = 0, 1/2, 1, 3/2, \dots$ in \hbar units). Notice that the last subset is labelled both by $j' \equiv \{j'_C\}$, $C = 1, 2, \dots, n'_1$, and by $m \equiv \{m_C\}$, where m_C is the projection of j'_C along the fixed reference axis (of course, for each m , $-j \leq m \leq j$ in integer steps). The consistency in the assignment of the j, j', m variables is ensured if we require that

- each 3-simplex σ_B^3 , $(B = 1, 2, \dots, N_3)$, in $(T^3, \partial T^3)$ must be associated, apart from a phase factor, with a $6j$ symbol of $SU(2)$, namely

$$\sigma_B^3 \longleftrightarrow (-1)^{\sum_{p=1}^6 j_p} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}_B; \quad (4)$$

- each 2-simplex σ_D^2 , $D = 1, 2, \dots, n'_2$ in ∂T^3 must be associated with a Wigner $3jm$ symbol of $SU(2)$ according to

$$\sigma_D^2 \longleftrightarrow (-1)^{(\sum_{s=1}^3 m_s)/2} \left(\begin{matrix} j'_1 & j'_2 & j'_3 \\ m_1 & m_2 & -m_3 \end{matrix} \right)_D. \quad (5)$$

Then the following state sum can be defined

$$\begin{aligned} Z_{PR}^3 &\equiv Z_{PR}[(M^3, \partial M^3)] = \\ &= \lim_{L \rightarrow \infty} \sum_{\left\{ \begin{matrix} (T^3, \partial T^3) \\ j, j', m \leq L \end{matrix} \right\}} Z[(T^3(j), \partial T^3(j', m)) \rightarrow (M^3, \partial M^3); L], \end{aligned} \quad (6)$$

where

$$\begin{aligned}
Z[(T^3(j), \partial T^3(j', m)) \rightarrow (M^3, \partial M^3); L] = \\
= \Lambda(L)^{-N_0} \prod_{A=1}^{N_1} (-1)^{2j_A} (2j_A + 1) \prod_{B=1}^{N_3} (-1)^{\sum_{p=1}^6 j_p} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}_B \cdot \\
\cdot \prod_{D=1}^{n'_2} (-1)^{(\sum_{s=1}^3 m_s)/2} \left(\begin{matrix} j'_1 & j'_2 & j'_3 \\ m_1 & m_2 & -m_3 \end{matrix} \right)_D. \tag{7}
\end{aligned}$$

N_0, N_1, N_3 denote respectively the total number of vertices, edges and tetrahedra in $(T^3(j), \partial T^3(j', m))$, while n'_2 is the number of 2-simplices lying in $\partial T^3(j', m)$. Notice that there appears a factor $\Lambda(L)^{-1}$ for each vertex in $\partial T^3(j', m)$, with $\Lambda(L) \equiv 4L^3/3C$, C an arbitrary constant.

The state sum given in (6) and (7) when $\partial M^3 = \emptyset$ reduces to the usual Ponzano–Regge partition function for a closed manifold M^3 ; in such a case, it can be rewritten simply as

$$Z_{PR}[M^3] = \lim_{L \rightarrow \infty} \sum_{\{T^3(j), j \leq L\}} Z[T^3(j) \rightarrow M^3; L], \tag{8}$$

where the sum is extended to all assignments of spin variables such that each of them is not greater than the cut-off L , and each term under the sum is given by

$$Z[T^3(j) \rightarrow M^3; L] = \Lambda(L)^{-N_0} \prod_{A=1}^{N_1} (-1)^{2j_A} (2j_A + 1) \prod_{B=1}^{N_3} (-1)^{\sum_{p=1}^6 j_p} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}_B. \tag{9}$$

As is well known, the above state sum gives the semiclassical partition function of Euclidean 3-gravity with an action discretized according to Regge's prescription [14]. Moreover, it is formally invariant under any finite set of topological transformations performed on 3-simplices in $T^3(j)$: following Pachner [9], they are commonly known as *bistellar moves*. It is a classical result (see *e.g.* [12] and [4]) that such moves can be expressed algebraically in terms of the Biedenharn–Elliott identity (representing the moves (2 tetrahedra) \leftrightarrow (3 tetrahedra)) and of both the B-E identity and the orthogonality conditions for $6j$ symbols, which represent the barycentric move together with its inverse, namely (1 tetrahedron) \leftrightarrow (4 tetrahedra) (see [19] for the explicit expressions of these identities as well as for notations concerning other (re)coupling coefficients).

In general, if we denote by n_d the number of d -simplices $\in T^d$ involved in a given bistellar operation, then such a move can be represented with the notation

$$[n_d \rightarrow (d+1) - (n_d - 1)]_{bst}^{\mathbf{d}} \tag{10}$$

and the entire set of allowed moves in dimension d is found for $n_d = 1, 2, \dots, d+1$ (as an example, the barycentric subdivision corresponds to the case $n_d = 1$).

The invariance under bistellar moves is related to the PL -equivalence class of the manifolds involved. Indeed, Pachner proved in [9] that two closed d -dimensional PL -manifolds are PL -homeomorphic if, and only if, their underlying triangulations are related to each other by a finite sequence of bistellar moves. Thus in particular the state sum (8) is formally an invariant of the PL -structure of M^3 (the regularized counterpart being the Turaev–Viro invariant found in [18]).

Turning now to the non trivial case of (6) with $\partial M^3 \neq \emptyset$, new types of topological transformations have to be taken into account. Indeed Pachner introduced moves which are suitable in the case of compact d -dimensional PL -manifolds with a non-empty boundary, the *elementary shellings* (see [10]). This kind of operation involves the cancellation of one d -simplex at a time in a given triangulation $(T^d, \partial T^d) \rightarrow (M^d, \partial M^d)$ of a compact PL -pair of dimension d . In order to be deleted, the d -simplex must have some of its $(d-1)$ -dimensional faces lying in the boundary ∂T^d . Moreover, for each elementary shelling there exists an inverse move which corresponds to the attachment of a new d -simplex to a suitable component in ∂T^d . It is possible to classify the two sets of moves by setting

$$[n_{d-1} \rightarrow d - (n_{d-1} - 1)]_{sh}^d, \quad [n_{d-1} \rightarrow d - (n_{d-1} - 1)]_{ish}^d, \quad (11)$$

where n_{d-1} represents the number of $(d-1)$ -simplices (belonging to a single d -simplex) involved in an elementary shelling and in an inverse shelling, respectively. Then the full set of operations is found when n_{d-1} runs over $(1, 2, \dots, d)$ in both cases.

In [2] identities representing the three types of elementary shellings (and their inverse moves) for the 3-dimensional triangulation given in (3) were established.

The first identity represents, according to (11), the move $[2 \rightarrow 2]_{sh}^3$. The topological content of this identity is drawn on the top of FIG.1, while its formal expression reads

$$\begin{aligned} \sum_{c\gamma} (2c+1)(-1)^{2c-\gamma} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} c & r & p \\ -\gamma & \rho' & \psi \end{pmatrix} (-1)^\Phi \begin{Bmatrix} a & b & c \\ r & p & q \end{Bmatrix} = \\ = (-1)^{-2\rho} \sum_{\kappa} (-1)^{-\kappa} \begin{pmatrix} p & a & q \\ \psi & \alpha & -\kappa \end{pmatrix} \begin{pmatrix} q & b & r \\ \kappa & \beta & -\rho' \end{pmatrix}, \end{aligned} \quad (12)$$

where Latin letters a, b, c, r, p, q, \dots denote angular momentum variables, Greek letters $\alpha, \beta, \gamma, \rho, \psi, \kappa, \dots$ are the corresponding momentum projections and $\Phi \equiv a + b + c + r + p + q$.

Notice that in this section we agree that all j variables appearing in $3jm$ symbols are associated with edges lying in ∂T^3 in a given configuration, while j arguments of the $6j$ may belong either to ∂T^3 (if they have a counterpart in the nearby $3jm$) or to $int(T^3)$.

The other identities can be actually derived (up to suitable regularization factors) from (12) and from both the orthogonality conditions for the $6j$ symbols and the

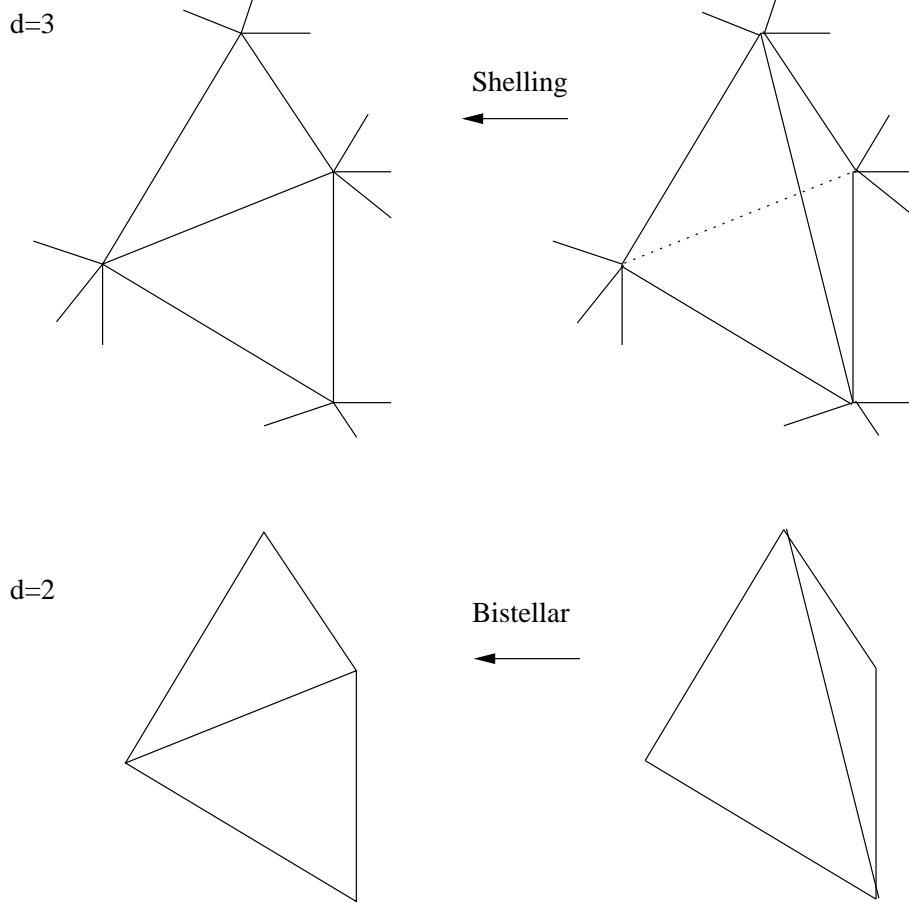


Figure 1: The shelling $[2 \rightarrow 2]_{sh}^3$ and the corresponding bistellar $[2 \rightarrow 2]_{bst}^2$.

completeness conditions for the $3jm$ symbols (see [19]). In particular, the shelling $[1 \rightarrow 3]_{sh}^3$ is sketched on the top of FIG.2 and the corresponding identity is given by

$$\begin{aligned} & \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} (-1)^\Phi \left\{ \begin{matrix} a & b & c \\ r & p & q \end{matrix} \right\} = \\ & = \sum_{\kappa\psi\rho} (-1)^{-\psi-\kappa-\rho} \begin{pmatrix} p & a & q \\ \psi & \alpha & -\kappa \end{pmatrix} \begin{pmatrix} q & b & r \\ \kappa & \beta & -\rho \end{pmatrix} \begin{pmatrix} r & c & p \\ \rho & \gamma & -\psi \end{pmatrix}. \quad (13) \end{aligned}$$

Finally, the shelling $[3 \rightarrow 1]_{sh}^3$ is depicted on the top of FIG.3 and the associated identity reads

$$\begin{aligned} & \Lambda(L)^{-1} \sum_{q\kappa', p\psi', r\rho'} (-1)^{-\psi'-\kappa'-\rho'} (-1)^{2(p+q+r)} (2p+1)(2r+1)(2q+1) \cdot \\ & \cdot \begin{pmatrix} a & p & q \\ \alpha & -\psi' & \kappa' \end{pmatrix} \begin{pmatrix} b & q & r \\ \beta & -\kappa' & \rho' \end{pmatrix} \begin{pmatrix} c & r & p \\ \gamma & -\rho' & \psi' \end{pmatrix} (-1)^\Phi \left\{ \begin{matrix} a & b & c \\ r & p & q \end{matrix} \right\} = \end{aligned}$$

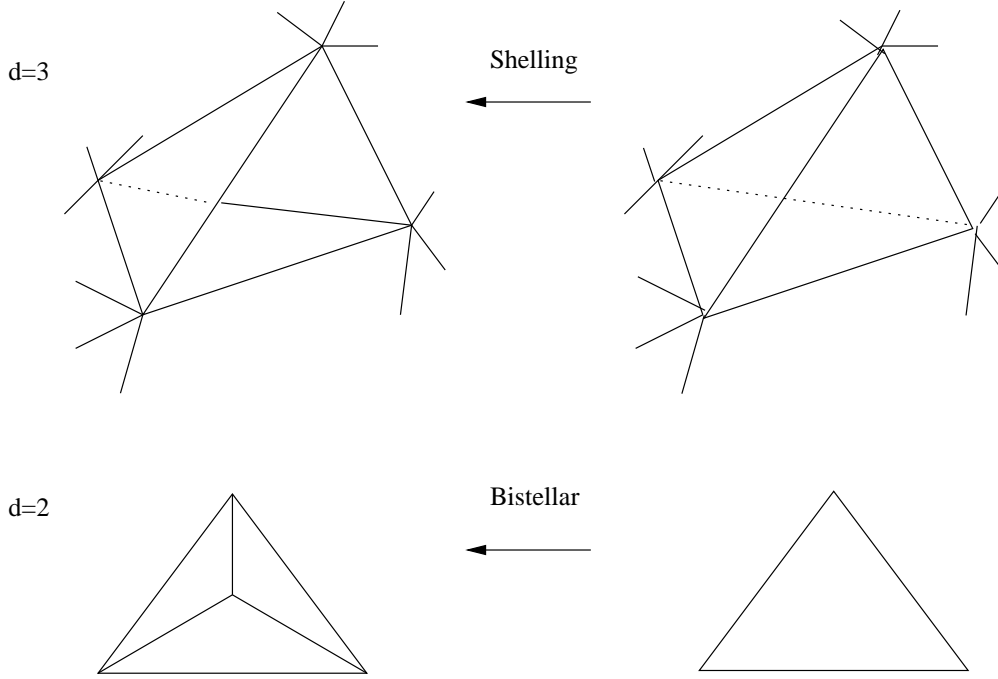


Figure 2: The shelling $[1 \rightarrow 3]_{sh}^3$ and the corresponding bistellar $[1 \rightarrow 3]_{bst}^2$.

$$= \begin{pmatrix} b & a & c \\ \beta & \alpha & \gamma \end{pmatrix}, \quad (14)$$

where $\Lambda(L)$ is defined as in (7).

Notice that in each of the above identities we can read also the corresponding inverse shelling, namely the operation consisting in the attachment of a 3-simplex to the suitable component(s) in ∂T^3 , simply by exchanging the role of internal and external labellings.

Comparing the above identities representing the elementary shellings and their inverse moves with the expression given in (7), we see that the state sum $Z_{PR}[(M^3, \partial M^3)]$ in (6) is formally invariant both under (a finite number of) bistellar moves in the interior of M^3 , and under (a finite number of) elementary boundary operations. Following now [10] we are able to conclude that (6) is indeed an invariant of the PL -structure (as well as a topological invariant, since we are dealing with 3-dimensional PL -manifolds).

Since the structure of a local arrangement of 2-simplices in the state sum (6) is naturally encoded in (12), (13) and (14), it turns out that a state sum for a 2-dimensional triangulation of a closed PL -manifold M^2

$$T^2(j; m, m') \longrightarrow M^2 \quad (15)$$

can be consistently defined if we require that

- each 2-simplex $\sigma^2 \in T^2$ is associated with the following product of two Wigner symbols (a *double* $3jm$ symbol for short)

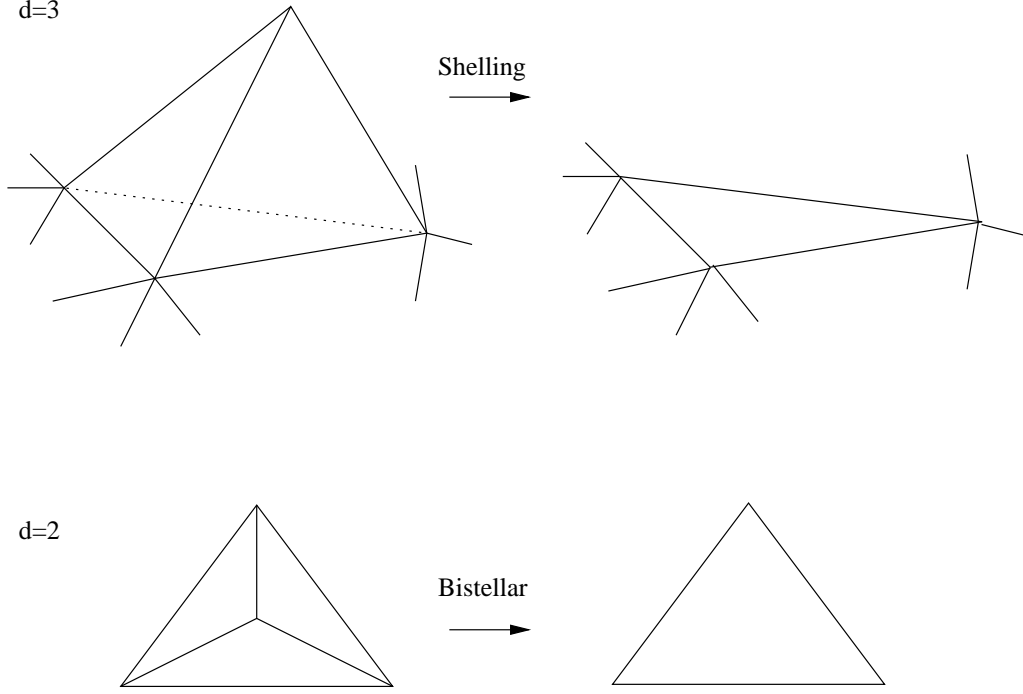


Figure 3: The shelling $[3 \rightarrow 1]_{sh}^3$ and the corresponding bistellar $[3 \rightarrow 1]_{bst}^2$.

$$\sigma^2 \longleftrightarrow (-1)^{\sum_{s=1}^3 (m_s + m'_s)/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & -m'_3 \end{pmatrix}, \quad (16)$$

where $\{m_s\}$ and $\{m'_s\}$ are two different sets of momentum projections associated with the same angular momentum variables $\{j_s\}$, $-j \leq m_s, m'_s \leq j \forall s = 1, 2, 3$. The expression of the state sum proposed in [3] reads

$$\begin{aligned} Z[T^2(j; m, m') \rightarrow M^2; L] &= \\ &= \Lambda(L)^{-N_0} \prod_{A=1}^{N_1} (2j_A + 1) (-1)^{2j_A} (-1)^{-m_A - m'_A} \\ &\quad \prod_{B=1}^{N_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}_B \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & -m'_3 \end{pmatrix}_B, \end{aligned} \quad (17)$$

where N_0, N_1, N_2 are the numbers of vertices, edges and triangles in T^2 , respectively. Summing over all of the admissible assignments of $\{j; m, m'\}$ we get

$$Z[M^2] = \lim_{L \rightarrow \infty} \sum_{\{T^2(j; m, m'), j \leq L\}} Z[T^2(j; m, m') \rightarrow M^2; L], \quad (18)$$

where the regularization is carried out according to the usual prescription. The invariance of (18) is ensured as far as the bistellar moves in $d = 2$ can be

implemented. One of these move is expressed according to

$$\begin{aligned}
& \sum_q \sum_{\kappa, \kappa'} (2q+1) (-1)^{2q} (-1)^{-\kappa-\kappa'} \begin{pmatrix} p & a & q \\ \psi & \alpha & -\kappa \end{pmatrix} \begin{pmatrix} q & b & r \\ \kappa & \beta & \rho \end{pmatrix} \\
& \cdot \begin{pmatrix} p & a & q \\ \psi' & \alpha' & -\kappa' \end{pmatrix} \begin{pmatrix} q & b & r \\ \kappa' & \beta' & \rho' \end{pmatrix} = \sum_c \sum_{\gamma, \gamma'} (2c+1) (-1)^{2c} (-1)^{-\gamma-\gamma'} \cdot \\
& \cdot \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} r & p & c \\ \rho & \psi & -\gamma \end{pmatrix} \begin{pmatrix} a & b & c \\ \alpha' & \beta' & \gamma' \end{pmatrix} \begin{pmatrix} r & p & c \\ \rho' & \psi' & -\gamma' \end{pmatrix} \quad (19)
\end{aligned}$$

and represents the so called *flip*, namely the bistellar move $[2 \rightarrow 2]_{bst}^2$, having taken into account the notation introduced in (10) (refer to the bottom of FIG.1 for the corresponding picture).

The identity corresponding to the remaining moves, namely $[1 \leftrightarrow 3]_{bst}^2$, reads

$$\begin{aligned}
& \sum_{q,r,p} (2q+1)(2r+1)(2p+1) (-1)^{2q+2r+2p} \sum_{\kappa, \kappa'} \sum_{\rho, \rho'} \sum_{\psi, \psi'} (-1)^{-\kappa-\kappa'} (-1)^{-\rho-\rho'} \cdot \\
& \cdot (-1)^{-\psi-\psi'} \begin{pmatrix} p & a & q \\ \psi & \alpha & -\kappa \end{pmatrix} \begin{pmatrix} q & b & r \\ \kappa & \beta & -\rho \end{pmatrix} \begin{pmatrix} r & c & p \\ \rho & \gamma & -\psi \end{pmatrix} \cdot \\
& \cdot \begin{pmatrix} p & a & q \\ \psi' & \alpha' & -\kappa' \end{pmatrix} \begin{pmatrix} q & b & r \\ \kappa' & \beta' & -\rho' \end{pmatrix} \begin{pmatrix} r & c & p \\ \rho' & \gamma' & -\psi' \end{pmatrix} = \\
& = \Lambda(L)^{-1} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} a & b & c \\ \alpha' & \beta' & \gamma' \end{pmatrix} \quad (20)
\end{aligned}$$

and these moves are depicted on the bottoms of FIG.2 and FIG.3.

As a matter of fact, the state sum given in (17) and (18) is formally invariant under (a finite number of) topological operations represented by (19) and (20). Thus, from Pachner's theorem proved in [9], we conclude that it is a (PL) topological invariant. Its expression can be easily evaluated also in the q -case, providing us with a finite quantum invariant given by

$$Z_\chi^2(q) \equiv Z[M^2](q) = w_q^2 w_q^{-2\chi(M^2)}, \quad (21)$$

where $\chi(M^2)$ is the Euler characteristic of the manifold M^2 and $w_q^2 = -2k/(q-q^{-1})^2$ (see *Appendix B*).

In conclusion, the 2-dimensional closed model generated by the 3-dimensional model with a non empty boundary is not trivial, being the only topological invariant which is significant for a closed surface in the present context.

3 Extension of the Crane–Yetter model to $(M^4, \partial M^4)$ and induced 3-dimensional invariant

In this section we revise first the results found in [5] (see also [8] and [20]) concerning the q -invariant $Z_{CY}[M^4](q)$ for a closed PL -manifold M^4 . However, for the sake of simplicity, we limit ourselves to a detailed analysis of the $(q = 1)$ case, and moreover we write down the expression of the resulting $Z_{CY}[M^4] \doteq Z_{CY}[M^4](q)|_{q=1}$ in terms of the $3jm$ symbols appearing in the definition of the $SU(2)$ $15j$ symbol of the second type (rather than using its expression in terms of $6j$ coefficients). This last step turns out to be crucial in order to define the new invariant $Z_{CY}[(M^4, \partial M^4)]$ for a PL -pair $(M^4, \partial M^4)$ and also in order to show that the state sum induced on $\partial M^4 \equiv M^3$ is indeed the topological invariant $Z_\chi[M^3]$ (which is trivial in the present case since M^3 is a closed manifold).

Thus, consider a *multi-colored* triangulation of a given closed PL -manifold M^4 denoting it by the map

$$T^4(j_{\sigma^2}, J_{\sigma^3}) \longrightarrow M^4, \quad (22)$$

where j_{σ^2} is an $SU(2)$ -coloring on the 2-dimensional simplices σ^2 in T^4 and J_{σ^3} is an $SU(2)$ -coloring on tetrahedra $\sigma^3 \subset \sigma^4 \in T^4$ (recall that an ordering on the vertices of each 4-simplex σ^4 has to be chosen; however, the final expression of the state sum turns out to be independent of this choice). The consistency in the assignment of the $\{j, J\}$ spin variables is ensured for a fixed ordering if we require that

- each 3-simplex $\sigma_a^3 \subset \sigma^4$ ($a = 1, 2, \dots, N_3$, N_3 being the number of 3-simplices in T^4) must be associated, apart from a phase factor, with a product of two $3jm$ symbols, namely

$$\sigma_a^3 \longleftrightarrow \begin{pmatrix} j_1 & j_2 & J_a \\ m_1 & m_2 & m_a \end{pmatrix} \begin{pmatrix} j_3 & j_4 & J_a \\ m_3 & m_4 & -m_a \end{pmatrix}; \quad (23)$$

- each 4-simplex $\sigma^4 \in T^4$ must be associated, apart from a phase factor, with a summation of the product of ten suitable $3jm$ symbols (see below for its explicit expression), giving rise to a $15j$ symbol of the second type which we represent for short as

$$\sigma^4 \longleftrightarrow [J_a, J_b, J_c, J_d, J_e]_{\sigma^4}, \quad (24)$$

where J_a, \dots, J_e are labellings assigned to the five tetrahedra $\sigma_a, \dots, \sigma_e \subset \sigma^4$.

Then we can define the following state sum

$$\begin{aligned}
Z[T^4(j_{\sigma^2}, J_{\sigma^3}) \rightarrow M^4; L] &= \\
&= \Lambda(L)^{N_0 - N_1} \prod_{\sigma^2 \in T^4} (-1)^{2j_{\sigma^2}} (2j_{\sigma^2} + 1) \prod_{\sigma^3 \in T^4} (-1)^{2J_{\sigma^3}} (2J_{\sigma^3} + 1) \cdot \\
&\cdot \prod_{\sigma^4 \in T^4} [J_a, J_b, J_c, J_d, J_e]_{\sigma^4},
\end{aligned} \tag{25}$$

where N_0, N_1 are the number of vertices and edges in T^4 , respectively. The $15j$ symbol associated with each 4-simplex is given explicitly by

$$\begin{aligned}
[J_a, J_b, J_c, J_d, J_e]_{\sigma^4} &\doteq \{15j\}_{\sigma^4}(J) = \\
&= \sum_m (-1)^{\sum m} \begin{pmatrix} j_1 & j_2 & J_a \\ m_1 & m_2 & m_a \end{pmatrix} \begin{pmatrix} j_3 & j_4 & J_a \\ m_3 & m_4 & -m_a \end{pmatrix} \begin{pmatrix} j_5 & j_6 & J_b \\ m_5 & m_6 & m_b \end{pmatrix} \cdot \\
&\cdot \begin{pmatrix} j_3 & j_7 & J_b \\ -m_3 & m_7 & -m_b \end{pmatrix} \begin{pmatrix} j_5 & j_8 & J_c \\ -m_5 & m_8 & m_c \end{pmatrix} \begin{pmatrix} j_1 & j_9 & J_c \\ -m_1 & m_9 & -m_c \end{pmatrix} \cdot \\
&\cdot \begin{pmatrix} j_6 & j_{10} & J_d \\ -m_6 & m_{10} & -m_d \end{pmatrix} \begin{pmatrix} j_2 & j_8 & J_d \\ -m_2 & -m_8 & -m_d \end{pmatrix} \begin{pmatrix} j_7 & j_{10} & J_e \\ -m_7 & -m_{10} & -m_e \end{pmatrix} \cdot \\
&\cdot \begin{pmatrix} j_4 & j_9 & J_e \\ -m_4 & -m_9 & -m_e \end{pmatrix},
\end{aligned} \tag{26}$$

where the summation is extended to all admissible values of the m variables, and the planar diagram corresponding to the symbol is sketched in FIG.4.

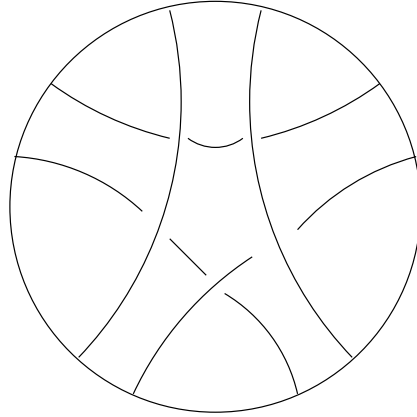


Figure 4: Diagrammatic representation of the $15j$ symbol of the second type.

It can be shown (see [5], [15]) that the expression

$$Z_{CY}[M^4] = \lim_{L \rightarrow \infty} \sum_{\left\{ \begin{array}{l} T^4(j, J) \\ j, J \leq L \end{array} \right\}} Z[T^4(j_{\sigma^2}, J_{\sigma^3}) \rightarrow M^4; L] \tag{27}$$

is formally a PL -invariant, and that its value is given by $\Lambda(L)^{\chi(M^4)/2} K^{\sigma(M^4)}$, where $\chi(M^4)$, $\sigma(M^4)$ are the Euler characteristic and the signature of M^4 respectively, and K is a constant.

The former state sum can be generalized to the case of a compact 4-dimensional PL -pair by considering the map

$$(T^4(j_{\sigma^2}, J_{\sigma^3}), \partial T^4(j'_{\sigma^2}, J'_{\sigma^3}; m_{\sigma^2}, m_{\sigma^3})) \longrightarrow (M^4, \partial M^4), \quad (28)$$

where $\{j_{\sigma^2}, J_{\sigma^3}\}$ denotes the entire set of spin variables, ranging from 1 to N_2 and from 1 to N_3 , respectively. The subset $\{j'_{\sigma^2}, J'_{\sigma^3}\} \subset \{j_{\sigma^2}, J_{\sigma^3}\}$ contains the colorings of the subsimplices in ∂T^4 , the corresponding magnetic numbers of which can be collectively denoted by $m \equiv \{m_{\sigma^2}, m_{\sigma^3}\}$ as far as no confusion arises. The assignment of the above variables turns out to be consistent if we agree with the statements in (24) (for all $\sigma^4 \in (T^4, \partial T^4)$, taking into account the fact that some of the J labels may become J' for those 4-simplices which have component(s) in ∂T^4) and with (23) (for 3-simplices in the interior of the triangulation). Moreover, we require that

- each 3-simplex $\sigma_a^3 \subset \sigma^4$ lying in the boundary ∂T^4 must be associated, apart from a phase factor, with the following product of $3jm$ symbols

$$\sigma_a^3 \longleftrightarrow \begin{pmatrix} j'_1 & j'_2 & J'_a \\ m_1 & m_2 & m_a \end{pmatrix} \begin{pmatrix} j'_3 & j'_4 & J'_a \\ m_3 & m_4 & -m_a \end{pmatrix}. \quad (29)$$

With these premises, we consider now the following expression

$$\begin{aligned} & Z[(M^4, \partial M^4)] = \\ & = \lim_{L \rightarrow \infty} \sum \left\{ \begin{array}{c} (T^4, \partial T^4) \\ j, J, j', J'; \\ m \leq L \end{array} \right\} Z[(T^4(j, J), \partial T^4(j', J'; m)) \rightarrow (M^4, \partial M^4); L], \quad (30) \end{aligned}$$

where we have used a shorthand notation instead of (28), and where

$$\begin{aligned} & Z[(T^4(j, J), \partial T^4(j', J'; m)) \rightarrow (M^4, \partial M^4); L] = \\ & = \Lambda(L)^{N_0 - N_1} \prod_{all \sigma^2} (-1)^{2j_{\sigma^2}} (2j_{\sigma^2} + 1) \prod_{all \sigma^3} (-1)^{2J_{\sigma^3}} (2J_{\sigma^3} + 1) \cdot \\ & \cdot \prod_{all \sigma^4} \{15j\}_{\sigma^4}(J, J') \prod_{\sigma^3 \in \partial T^4} (-1)^{\sum m_{j'} / 2 + \sum m_{J'}} \begin{pmatrix} j'_1 & j'_2 & J'_a \\ m_1 & m_2 & m_a \end{pmatrix} \cdot \\ & \cdot \begin{pmatrix} j'_3 & j'_4 & J'_a \\ m_3 & m_4 & -m_a \end{pmatrix}. \quad (31) \end{aligned}$$

Here we introduced explicitly in the phase factors $m_{j'}$ and $m_{J'}$ to denote magnetic numbers corresponding to different kinds of spin variables on the boundary.

As discussed in *Section 2*, for a *PL*-pair in dimension d there exist d different types of elementary shellings (and d inverse shellings), parametrized by the number n_{d-1} of faces in a boundary d -simplex according to (11). Thus in the present case we are dealing with four different shellings, $n_3 = 1, 2, 3, 4$ being the number of tetrahedra in ∂T^4 which are going to disappear (together with the underlying 4-simplex), respectively. The diagrammatic representations of the moves $[1 \rightarrow 4]_{sh}^4$, $[2 \rightarrow 3]_{sh}^4$, $[3 \rightarrow 2]_{sh}^4$, $[4 \rightarrow 1]_{sh}^4$ are displayed in FIG.5, FIG.6, FIG.7, FIG.8,

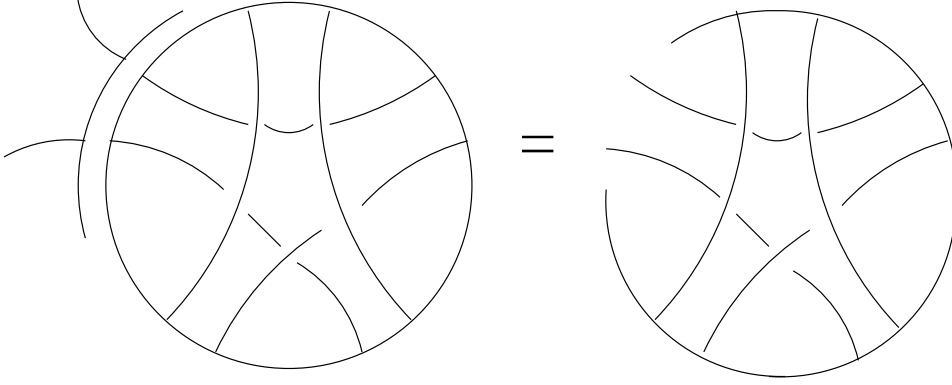


Figure 5: Diagrammatic representation of the move $[1 \rightarrow 4]_{sh}^4$.

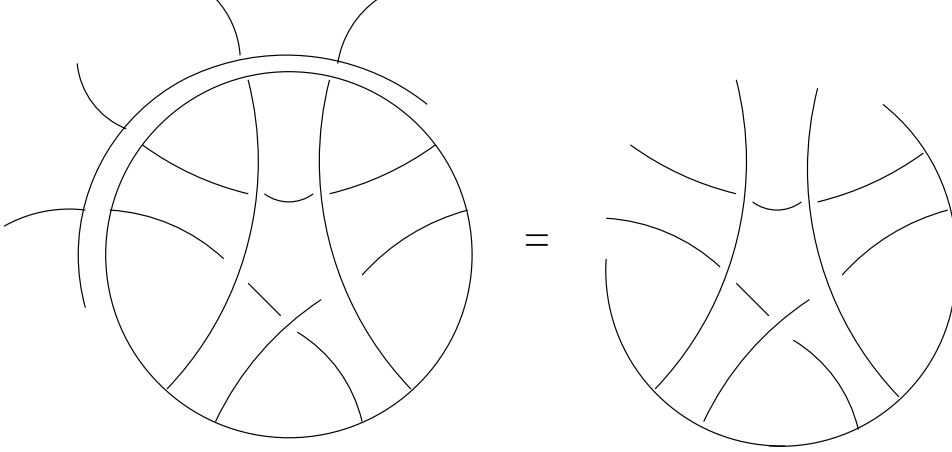


Figure 6: Diagrammatic representation of the move $[2 \rightarrow 3]_{sh}^4$.

respectively, where we have made use of diagrammatical relations to handle products of $3jm$ symbols (see FIG.9 and FIG.10). The explicit expressions of the algebraic identities associated with elementary shellings are collected in *Appendix A*. According to these remarks, $Z[(M^4, \partial M^4)]$ in (30) turns out to be formally equivalent under the action of a finite set of the above operations and their inverse moves and thus, owing to the theorem proved in [10], it defines an invariant of the *PL*-structure. Its q -deformed counterpart, $Z[(M^4, \partial M^4)](q)$, can be worked out accord-

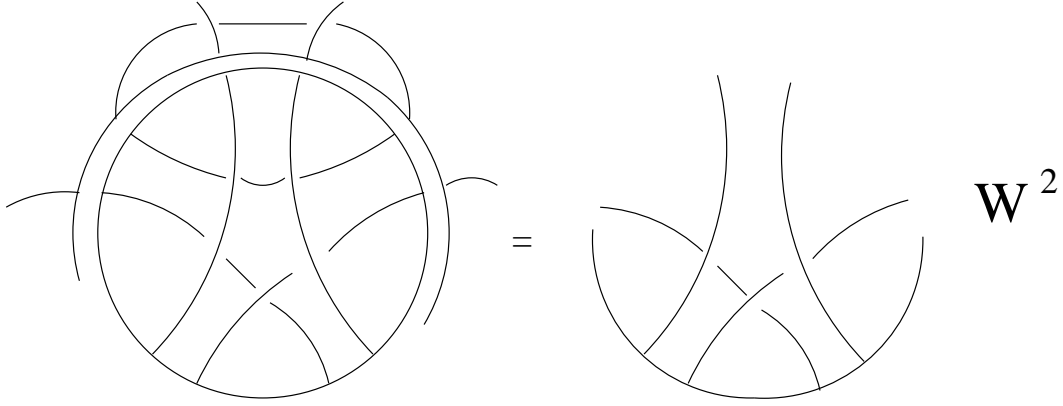


Figure 7: Diagrammatic representation of the move $[3 \rightarrow 2]_{sh}^4$.

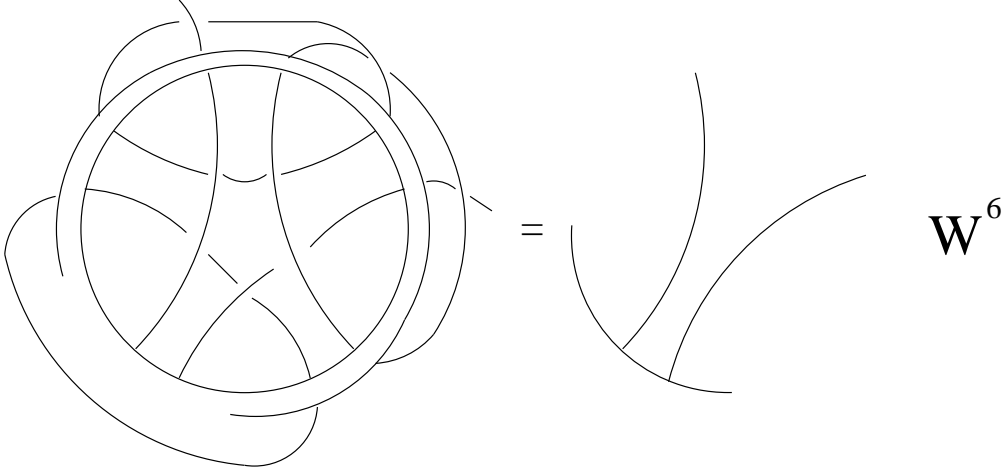


Figure 8: Diagrammatic representation of the move $[4 \rightarrow 1]_{sh}^4$.

ing to the prescription given in *Appendix B* and represents a well-defined quantum invariant.

We may notice also that, since (30) reduces to (27) when $\partial M^4 = \emptyset$, $Z[(M^4, \partial M^4)]$ is invariant under bistellar moves performed in $\text{int}(T^4)$ too. However, we will show in *Section 6* that the equivalence of our d -dimensional $Z[(M^d, \partial M^d)]$ under elementary shellings implies (in a non trivial way) the invariance under bistellar moves of the state sum induced by setting $\partial M^d = \emptyset$.

Looking now at the local arrangement of 3-simplices in the state sum (31), it turns out that an *induced* 3-dimensional state sum for a colored closed triangulation

$$T^3(j_{\sigma^2}, J_{\sigma^3}; m_{\sigma^2}, \mu_{\sigma^2}; m_{\sigma^3}, \mu_{\sigma^3}) \longrightarrow M^3 \quad (32)$$

can be defined by associating with each 3-simplex the following sum of products of Wigner symbols

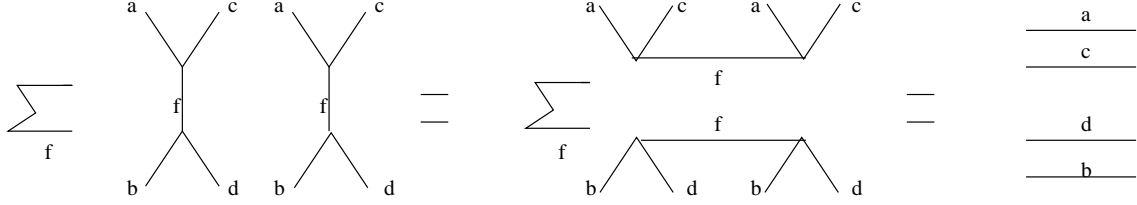


Figure 9: Relation involving the product of two couple of 3jm symbols.

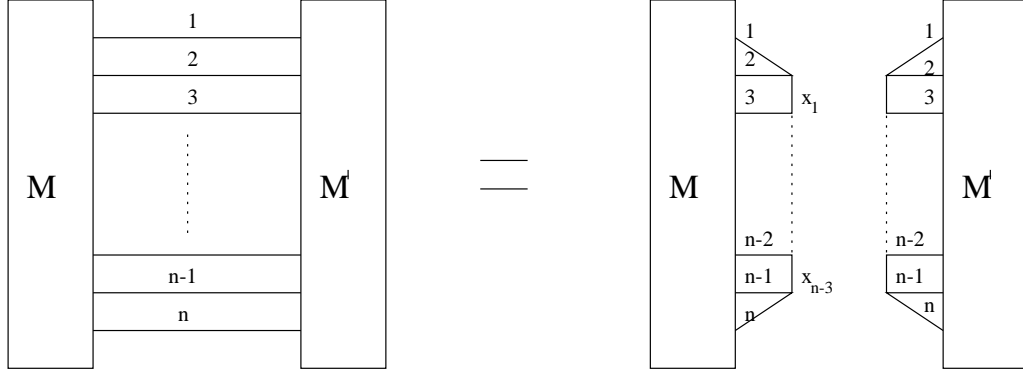


Figure 10: Diagrammatical representation of the general relation involving the product of two quantities containing many 3jm symbols; M and M' represent the portions of the diagram leaved unchanged by the relation.

$$\begin{aligned}
\sigma_a^3 \longleftrightarrow & \sum_{m_a, \mu_a} (-1)^{m_a + \mu_a} \begin{pmatrix} j_1 & j_2 & J_a \\ m_1 & m_2 & m_a \end{pmatrix} \begin{pmatrix} j_3 & j_4 & J_a \\ m_3 & m_4 & -m_a \end{pmatrix} \\
& \cdot \begin{pmatrix} j_1 & j_2 & J_a \\ \mu_1 & \mu_2 & \mu_a \end{pmatrix} \begin{pmatrix} j_3 & j_4 & J_a \\ \mu_3 & \mu_4 & -\mu_a \end{pmatrix} \doteq \\
& \doteq \begin{pmatrix} j_1 & j_2 & J_a & j_3 & j_4 \\ m_1 & m_2 & & m_3 & m_4 \\ \mu_1 & \mu_2 & & \mu_3 & \mu_4 \end{pmatrix}_{\sigma_a^3}. \tag{33}
\end{aligned}$$

Here all magnetic numbers $\{m, \mu\}$ have their natural ranges of variation with respect to the corresponding $\{j, J\}$ and a shorthand notation for the product of symbols has been introduced. According to (33), we can define the following state sum

$$\begin{aligned}
Z[T^3(j, J; m, \mu) \rightarrow M^3; L] = & \Lambda(L)^{N_0 - N_1} \prod_{all \sigma^2} (2j_{\sigma^2} + 1) \prod_{all \sigma_a^3} (2J_{\sigma_a^3} + 1) (-1)^{\sum (J+j)}. \\
& \cdot \begin{pmatrix} j_1 & j_2 & J_a & j_3 & j_4 \\ m_1 & m_2 & & m_3 & m_4 \\ \mu_1 & \mu_2 & & \mu_3 & \mu_4 \end{pmatrix}_{\sigma_a^3}, \tag{34}
\end{aligned}$$

which gives rise to the expression

$$Z[M^3] = \lim_{L \rightarrow \infty} \sum_{\left\{ \begin{array}{l} T^3(j, J, m, \mu) \\ j, J, m, \mu \leq L \end{array} \right\}} Z[T^3(j, J, m, \mu) \rightarrow M^3; L]. \quad (35)$$

At this point we should implement those topological operations which are suitable in the present context, namely the bistellar moves $[n_3 \rightarrow 5 - n_3]_{bst}^3$, $n_3 = 1, 2, 3, 4$. However, we may provide a straightforward proof that $Z[M^3]$ in (35) is related to the Euler characteristic of M^3 . The combination of symbols in each σ_a^3 (cfr. (33)) can be interpreted (putting the spin variables on the same foot) as a contribution of two triangles joined along J_a in a triangulation S^2 of a 2-dimensional surface uniquely associated with the given T^3 in (34). Then, by comparing such a structure with (16) and with Z_χ^2 in (18), we see that the contribution for a finite L of the sums over all 1-simplices (from 1 to N_1) and over all 2-simplices (from 1 to N_2) in S^2 amounts to $w_L^2 \cdot w_L^{2(N_1 - N_2)}$, where we set $\Lambda(L)^{-1} \equiv w_L^{-2}$. By considering again (33) and the fact that we are dealing with manifolds, it turns out that the number of 1-simplices in S^2 is related to N_2 and N_3 in T^3 by $N_1(S^2) = (N_2 + N_3)(T^3)$, and also that $N_2(S^2) = 2N_3(T^3)$. Thus, for each finite value of L , we would obtain $Z[M^3; L] = w_L^{2+2\chi(M^3)}$. The regularized version of this results reads

$$Z_\chi[M^3](q) = w_q^{2[1+\chi(M^3)]} \equiv w_q^2, \quad (36)$$

where we have taken into account the fact that the Euler characteristic vanishes for any 3-dimensional closed manifold.

4 Invariants of closed M^d from colorings of $(d-1)$ -simplices

An alternative way of defining the state sums which give rise to Z_χ^2 and Z_χ^3 comes out when we consider the relationships between integrals of products of Wigner D -functions and suitable products of $3jm$ symbols (see *e.g.* [19]). Picking up the 1-skeleton of the dual complex of a given triangulation (either T^2 or T^3) we can assign in a consistent way a $D_{m\mu}^j(R)$, $R \in SU(2)$, to each edge incident on the vertices of such graphs (obviously the vertices are 3-valent in $d=2$ and 4-valent in $d=3$). In this framework the role of the magnetic quantum numbers m, μ is made manifest by introducing the fat graph associated with each one of the former graphs: thus any edge acquires two sets of $SU(2)$ -colorings, namely $\{j, m\}$ and $\{j, \mu\}$. The next step consists in performing an integral over the R -variables of the product of the D -functions associated with the legs of the graph incident on each vertex. By collecting the terms generated by all vertices, we would get exactly the products of double $3jm$ symbols (with the correct phase factors) which appear in the expressions of both

Z_χ^2 and Z_χ^3 . The formal calculation can be easily translated in the diagrammatic language as shown in FIG.11.

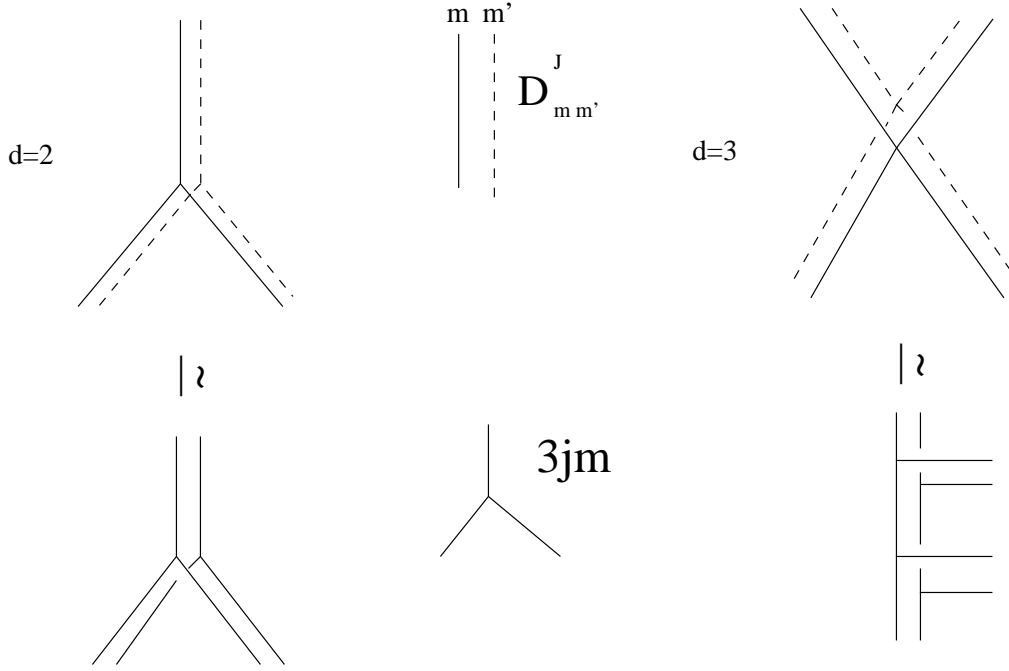


Figure 11: The graph representing the 1-skeleton of the dual lattice of each 2 and 3-dimensional simplex, with a D function associated with each edge, and the derived elementary diagrams occurring in Z_χ^2 and Z_χ^3 in terms of $3jm$ symbols.

The above procedure can be generalized to any dimension d . In particular, given any triangulation T^d of a closed PL -manifold M^d , we focus our attention on its dual 1-skeleton, which is a $(d+1)$ -valent graph Γ . As before, we are going to associate a D -function with each *fat* edge incident on each vertex. This amounts to require that consistent $SU(2)$ -colorings on Γ generated by the fat graph are achieved if we consider the assignment

$$(\{j\}; m, \mu) \longrightarrow \Gamma \subset \tilde{T}^d, \quad (37)$$

where \tilde{T}^d is the dual complex associated with T^d . The pair of magnetic numbers for each j variable refers to what we said before, namely we will have a double symbol, each component of which displays the same $\{j\}$ but different m, μ , for each elementary configuration of the dual graph.

Moreover, it should be clear that the assignments in (37) can be thought of as a coloring on the $(d-1)$ -skeleton of the original triangulation. According to this prescription, we can now define the following limit of admissible sums of configurations written in terms of $\{j\}$ variables

$$Z_\chi[M^d] = \lim_{L \rightarrow \infty} \sum_{\left\{ \begin{array}{l} T^d(\{j\}; m, \mu) \\ \{j\}, m, \mu \leq L \end{array} \right\}} w_L^{(-1)^{(d-1)\Psi}} \cdot \prod_{all \sigma^{d-1}} (-1)^{2j_{\sigma^{d-1}}} (2j_{\sigma^{d-1}} + 1) \left(\int_{\sigma^{d-1} \subset \sigma^d} D_{m\mu}^{j_{\sigma^{d-1}}}(R) dR \right)_{\sigma^d}, \quad (38)$$

where $w_L^2 = \Lambda(L)$ and $\Psi = 2(N_0 - N_1 + \dots + (-1)^{d-2}N_{d-2})$, (N_0, N_1, N_2, \dots) are the numbers of $(0, 1, 2, \dots)$ -simplices in T^d . The range of variation of each m, μ with respect to the corresponding j is the usual one, and the summations over the magnetic numbers act as glueing operations among labelled d -dimples.

Now we can exploit the relationships between integrals of products of D -functions and suitable combinations of Wigner symbols: this amounts to recognize that the integration in (38) can be recasted as

$$\begin{aligned} \sum_{\{J\}} Z[\sigma^d(\{j\}, \{J\}; m, \mu; \mathcal{M}, \nu) \rightarrow M^d; L] &\equiv \\ &\equiv \sum_{\{J\}} \prod_{k=1}^{d-3} (-1)^{2J_k} (2J_k + 1) \sum_{\mathcal{M}, \nu} (-1)^{\sum \mathcal{M} + \sum \nu} \begin{pmatrix} j_1 & j_2 & J_1 \\ m_1 & m_2 & \mathcal{M}_1 \end{pmatrix} \cdot \\ &\quad \begin{pmatrix} J_1 & j_3 & J_2 \\ -\mathcal{M}_1 & m_3 & \mathcal{M}_2 \end{pmatrix} \cdot \begin{pmatrix} j_1 & j_2 & J_1 \\ \mu_1 & \mu_2 & \nu_1 \end{pmatrix} \begin{pmatrix} J_1 & j_3 & J_2 \\ -\nu_1 & \mu_3 & \nu_2 \end{pmatrix} \cdots \\ &\quad \cdots \begin{pmatrix} J_{d-4} & j_{d-2} & J_{d-3} \\ -\mathcal{M}_{d-4} & m_{d-2} & \mathcal{M}_{d-3} \end{pmatrix} \begin{pmatrix} J_{d-3} & j_{d-1} & j_d \\ -\mathcal{M}_{d-3} & m_{d-1} & m_d \end{pmatrix} \cdot \\ &\quad \cdot \begin{pmatrix} J_{d-4} & j_{d-2} & J_{d-3} \\ -\nu_{d-4} & \mu_{d-2} & \nu_{d-3} \end{pmatrix} \begin{pmatrix} J_{d-3} & j_{d-1} & j_d \\ -\nu_{d-3} & \mu_{d-1} & \mu_d \end{pmatrix}. \end{aligned} \quad (39)$$

Here we put in evidence, besides the $\{j\}$ colorings, also the set $\{J\}$, the role of which is similar to what we found *e.g.* in (33), namely J variables come out to be associated with an internal glueing between the two sub-symbols of a double symbol. Moreover, we denoted by \mathcal{M}, ν the two sets of magnetic numbers associated with each J and by m, μ those associated with each j , respectively. The diagrammatic counterpart of the procedure described above is shown in FIG.12, where for simplicity just one of the possible coupling schemes is considered (the other ones giving equivalent analytical expressions).

Thus the limit in (38) can be rewritten as

$$Z[M^d] = \lim_{L \rightarrow \infty} \sum_{\left\{ \begin{array}{l} T^d(\{j\}, \{J\}; m, \mu) \\ all \ j, m \leq L \end{array} \right\}} w_L^{(-1)^{(d-1)\Psi}}.$$

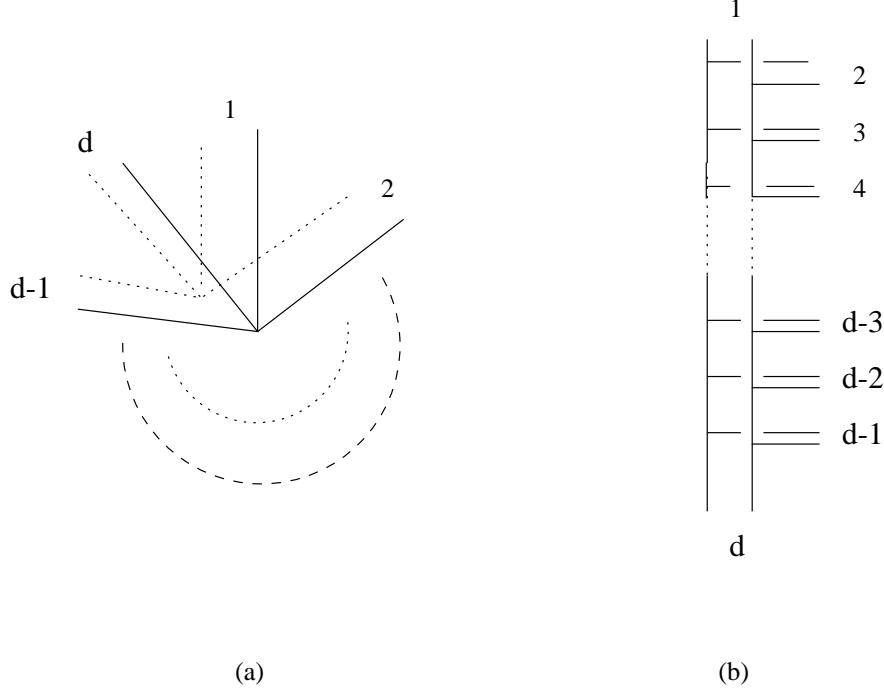


Figure 12: The graph representing the 1-skeleton of the dual lattice of each d -dimensional simplex, with a D function associated with each edge, and the derived elementary diagrams occurring in Z_χ^d in terms of 3jm symbols.

$$\prod_{\text{all } \sigma^{d-1}} (-1)^{2j_{\sigma^{d-1}}} (2j_{\sigma^{d-1}} + 1) \prod_{k=1}^{d-3} (-1)^{2J_k} (2J_k + 1) \sum_{\mathcal{M}, \nu} (-1)^{\sum \mathcal{M} + \sum \nu} \cdot Z[\sigma^d(\{j\}, \{J\}; m, \mu; \mathcal{M}, \nu) \rightarrow M^d; L], \quad (40)$$

where we used the shorthand notation defined in (39).

An explicit calculation which involves the auxiliary surface S^2 (*cfr.* the discussion at the end of *Section 3*), where now we get $(d-2)$ triangles associated with each σ^d in the original triangulation T^d , yields the regularized result

$$Z[M^d](q) = w_q^{2[1+(-1)^{(d-1)}\chi(M^d)]}, \quad (41)$$

where $\chi(M^d)$ is the Euler characteristic of M^d .

On passing, we may note that the counterpart of (41) in the continuum approach would be the d -dimensional topological field theory with a supersymmetric action given *e.g.* in [1].

5 Invariants of $(M^d, \partial M^d)$ induced by Z_χ^{d-1}

According to the program outlined in *steps 1), 2)* of the introduction, we build up in the following a state sum for a pair $(T^d, \partial T^d)$ induced by examining the expression

of $Z_\chi[\partial M^d \equiv M^{d-1}]$, with $Z_\chi[M^{d-1}]$ given by (39) of *Section 4*. The second part of the present section will be devoted to the proof that such state sums are actually independent of the triangulation chosen, and thus the invariant $Z[(M^d, \partial M^d)]$ is well defined in any dimension d .

We learnt from the procedure followed both in *Section 2* and in *Section 3* that once we give the *double* symbol associated with the $(d-1)$ -simplex in a given closed T^{d-1} , we can recover the contribution from a single d -simplex in $(T^d, \partial T^d \equiv T^{d-1})$ by taking first the *squared root* of the symbol itself. Then the recoupling symbol to be associated with the d -dimensional fundamental block is obtained by summing over the free m entries the product of $(d+1)$ -contributions from its faces, with suitable j labels. Thus in $d=2$ we used the symbol given in (16) as a fundamental block in the state sum $Z[T^2(j; m, m') \rightarrow M^2; L]$ (see (17)), while in $d=3$ the symbol given in (33) was associated with each 3-simplex of $T^3(j, J; m, \mu) \rightarrow M^3$ (see (34)): in both cases the double symbol looks like a square of some sub-symbol. Taking the squared root means that we pick up just one of the sub-symbols (*e.g.* either the Wigner symbol with m entries in (16) or the product of two Wigner symbols with m entries in (33)). By summing over all m entries the product of four sub-symbols with suitable j labellings and phase factors in $d=3$ we get the expression of the $6j$ symbol which enters $Z_{PR}[M^3]$. In turn, a summation over m variables of the product of five sub-symbols with suitable j labellings and phase factors in $d=4$ reproduces the $15j$ symbol which represents the fundamental block in $Z_{CY}[M^4]$ (see (26)). Notice also that the procedure works with PL -pairs as well: we simply associate one of the former sub-symbol (with an m' coloring, say) with each $(d-1)$ -simplex in ∂T^d .

The above remarks suggest how an algorithmic procedure for generating $Z[(M^d, \partial M^d)]$ from $Z_\chi[M^{d-1}]$ could be actually established. To this end, we consider first the structure of the double symbol associated with the fundamental block in (39), written for a closed triangulation T^{d-1} . The corresponding planar graph is shown in FIG.13 (compare also FIG.12): the diagram includes d external legs, representing



Figure 13: Diagram corresponding to the fundamental block occurring in the recoupling symbol associated with the d -dimensional simplex.

the faces of the $(d-1)$ -simplex, and $(d-3)$ internal edges, the colorings of which are associated with the $(d-1)$ -simplex itself, as explained below. The fundamental d -dimensional block which will enter the state sum is obtained by assembling $(d+1)$

remarks, turns out to be generic). A closer inspection of its combinatorial structure shows that we get in fact a $\{3(d-2)(d+1)/2\}j$ symbol written in terms of (sums of) $3jm$ symbols (see *e.g.* [21]). Collecting all the previous remarks we are now able to build up a state sum for $(T^d, \partial T^d)$ on the basis of the requirements listed below.

- 23

i) an admissible set of colorings on each of its $(d-2)$ -faces, namely j_1, j_2, \dots, j_F , where the value of the binomial coefficient

$$\binom{d+1}{2} \equiv F \quad (42)$$

gives the number of $(d-2)$ subsimplices of a d -simplex;

ii) other sets of $SU(2)$ -colorings associated with each of its $(d-1)$ -faces and denoted collectively by $\{J_{i_1}\}, \{J_{i_2}\}, \dots, \{J_{i_{d+1}}\}$ (these sets are the counterparts of the five labels J_a, \dots, J_e used in (24)). Then we set

$$\sigma^d \longleftrightarrow \left\{ \frac{3}{2}(d-2)(d+1)j \right\}_{\sigma^d} \doteq [\{J_{i_1}\}, \{J_{i_2}\}, \dots, \{J_{i_{d+1}}\}]_{\sigma^d} \quad (43)$$

- For each $\sigma^{d-1} \subset \partial T^d$, denoting as usual by $\{j', J'\} \subset \{j, J\}$ the subsets of spin variables belonging to boundary components, and labelling as J'_1, J'_2, \dots, J'_C ($C = d-3$) the variables associated with the internal legs of FIG.13, we have the explicit correspondence

$$\begin{aligned} \sigma^{d-1} \longleftrightarrow & \sum_{\mathcal{M}} (-1)^{\sum_{C=1}^{d-3} \mathcal{M}_C} \begin{pmatrix} j'_1 & j'_2 & J'_1 \\ m_1 & m_2 & \mathcal{M}_1 \end{pmatrix} \cdot \\ & \cdot \begin{pmatrix} J'_1 & j'_3 & J'_2 \\ -\mathcal{M}_1 & m_3 & \mathcal{M}_2 \end{pmatrix} \cdots \begin{pmatrix} J'_{d-4} & j'_{d-2} & J'_{d-3} \\ -\mathcal{M}_{d-3} & m_{d-2} & \mathcal{M}_{d-3} \end{pmatrix} \cdot \\ & \cdot \begin{pmatrix} J'_{d-3} & j'_{d-1} & j'_d \\ -\mathcal{M}_{d-3} & m_{d-1} & m_d \end{pmatrix}. \end{aligned} \quad (44)$$

Here we agree that each m variable is associated with the corresponding j' on the upper row, while an \mathcal{M} entry is the magnetic number of the upper J' , with the usual ranges of variations in both cases.

Then we can define the following state sum

$$\begin{aligned} Z[(T^d(j_{\sigma^{d-2}}, J_{\sigma^{d-1}}), \partial T^d(j'_{\sigma^{d-2}}, J'_{\sigma^{d-1}}; m, \mathcal{M}))] \rightarrow (M^d, \partial M^d; L) = \\ = w_L^{(-1)^{d\Xi}} \prod_{\text{all } \sigma^{d-2}} (-1)^{2j_{\sigma^{d-2}}(2j_{\sigma^{d-2}}+1)} \prod_{\text{all } \sigma^{d-1}} \left(\prod_{C=1}^{d-3} (-1)^{2J_C(2J_C+1)} \right)_{\sigma^{d-1}} \cdot \\ \cdot \prod_{\text{all } \sigma^d} \left\{ \frac{3}{2}(d-2)(d+1)j \right\}_{\sigma^d} (J, J') \prod_{\sigma^{d-1} \in \partial T^d} \sum_{\mathcal{M}} (-1)^{\sum m/2 + \sum \mathcal{M}} \cdot \\ \cdot \begin{pmatrix} j'_1 & j'_2 & J'_1 \\ m_1 & m_2 & \mathcal{M}_1 \end{pmatrix} \cdots \begin{pmatrix} J'_{d-3} & j'_{d-1} & j'_d \\ -\mathcal{M}_{d-3} & m_{d-1} & m_d \end{pmatrix}, \end{aligned} \quad (45)$$

where $\Xi = 2(N_0 - N_1 + \dots + (-1)^{d-3} N_{d-3})$, (N_0, N_1, N_2, \dots) being the total number of $(0, 1, 2, \dots)$ -dimensional simplices. Notice also that some of the recoupling coefficients associated with d -simplices may depend also on J' variables, if they have components in ∂T^d .

The limiting procedure for handling all state sums (45) for the given $(M^d, \partial M^d)$ can be defined as

$$Z[(M^d, \partial M^d)] = \lim_{L \rightarrow \infty} \sum_{\left\{ \begin{array}{l} T^d, \partial T^d \\ j, J \leq L \end{array} \right\}} Z[(T^d(j, J), \partial T^d(j', J'; m, \mathcal{M})) \rightarrow (M^d, \partial M^d); L], \quad (46)$$

where the ranges of the magnetic quantum numbers are $|m| \leq j', |\mathcal{M}| \leq J'$ in integer steps, and where suitable shorthand notations have been employed.

As anticipated before, we turn now to the basic question of the equivalence of $Z[(M^d, \partial M^d)]$ in (46) under a suitable class of topological operations. In the present case we are going to implement the set of elementary inverse shellings $[n_{d-1} \rightarrow d - (n_{d-1})]_{ish}^d$ in (11), involving the attachment of one d -simplex to ∂T^d along $n_{d-1} = 1, 2, \dots, d$ simplices of dimension $(d-1)$ glued together in suitable configurations. The complementary set of the elementary shellings could be singled out simply by exchanging internal and external labellings in a consistent way. It should be clear from similar discussions on equivalence in *Section 3*, *Section 4* and *Appendix A* that the explicit expressions of the identities associated with the moves become more and more complicated as dimension grows. Thus, we limit ourselves to the implementation of the moves through the diagrammatical method, which has been already used in *Section 4*. As a further remark, we note that glueing operations performed on triangulations underlying PL -pairs of manifolds (as happens in our context) involve joining of couples of p -dimensional simplices ($p = d, d-1$) along their *unique* common $(p-1)$ -dimensional face. More precisely, also the joining of two $(d-1)$ -dimensional simplices in ∂T^d has to fulfill this rule, since ∂T^d is a manifold (indeed, this would be true for pseudomanifolds as well).

Before dealing with the full d -dimensional case, let us illustrate the case of elementary inverse shellings in $d = 5$. Recall that a 5-simplex has six 4-dimensional simplices in its boundary and that in the present case n_{d-1} in (11) may run over 1, 2, 3, 4, 5. Consider first the inverse shelling $[5 \rightarrow 1]_{ish}^5$, the action of which is displayed in the diagram of FIG.15 (where we have made use of the diagrams shown in FIG.10 of *Section 3*). This operation amounts to glue a 5-simplex to ∂T^5 along five 4-simplices (joined among them along 3-dimensional faces). The resulting configuration in the modified $\partial T'^5$ gives rise to an unique new (open) 4-simplex, so that no new $(5-r)$ -simplices ($r \geq 3$) appear. Thus the state sum (45) does not acquire any w_L^{-2} factor and is manifestly invariant under such a move.

The inverse shelling $[4 \rightarrow 2]_{ish}^5$ consists in the attachment of a 5-simplex along four 4-simplices in ∂T^5 . In the new $\partial T'^5$ there appear two 4-simplices joined along a common 3-simplex (for what we said before), and thus also in this case we do not introduce new $(5-r)$ -dimensional subsimplices ($r \geq 3$) in the state sum and no additional w_L^{-2} factor arises. The diagrammatic proof is given in FIG.16, where we

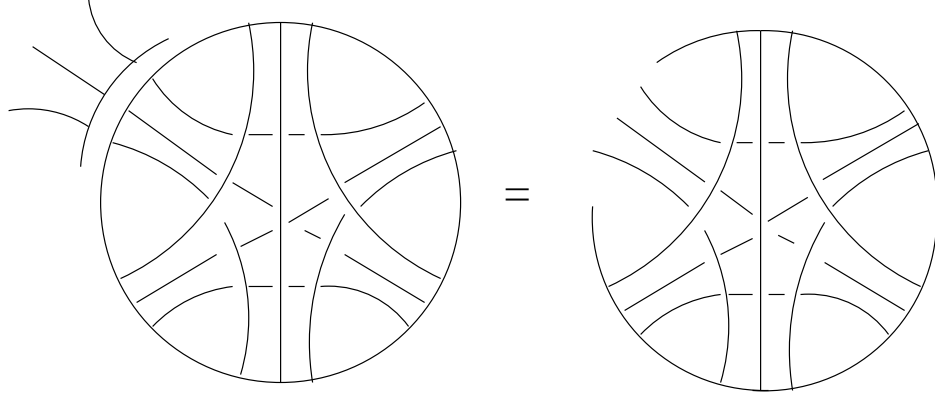


Figure 15: Shelling of $[5 \rightarrow 1]_{ish}^5$ type.

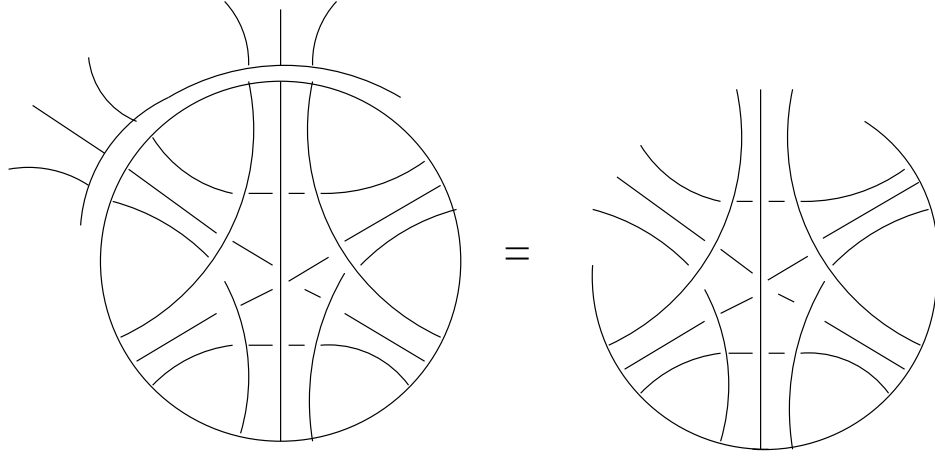


Figure 16: Shelling of $[4 \rightarrow 2]_{ish}^5$ type.

have taken into account FIG.10 again.

The $[3 \rightarrow 3]_{ish}^5$ move amounts to the glueing of a 5-simplex along three 4-simplices lying in ∂T^5 . Owing to the general remark that in a p -simplex $\subset \sigma^d$, exactly three $(p - 2)$ -dimensional subsimplices incide over a $(d - 3)$ -dimensional subsimplice, we see that such an inverse shelling generates just one new triangle in ∂T^5 (and no additional $(5 - r)$ -simplices with $r \geq 4$), associated with a w_L^{-2} factor in the state sum. The action of this move is reproduced in FIG.17, where we can see the loop bringing a w_L^2 factor which cancels the contribution coming from the new triangle. The move $[2 \rightarrow 4]_{ish}^5$ represents the attachment of a 5-simplex to ∂T^5 along four 4-dimensional simplices. Recalling the expressions giving the number of subsimplices of a p -simplex (see below), we may see that the configuration of two 4-simplices, glued along their common 3-dimensional face, identifies six vertices, fourteen edges, sixteen triangles and nine tetrahedra; thus this type of inverse shelling generates in ∂T^5 one new edge ($(d - 4)$ -simplex) and four triangles ($(d - 3)$ -simplices), giving rise to w_L^{-6} factor in the state sum. The above action is depicted in the diagram of

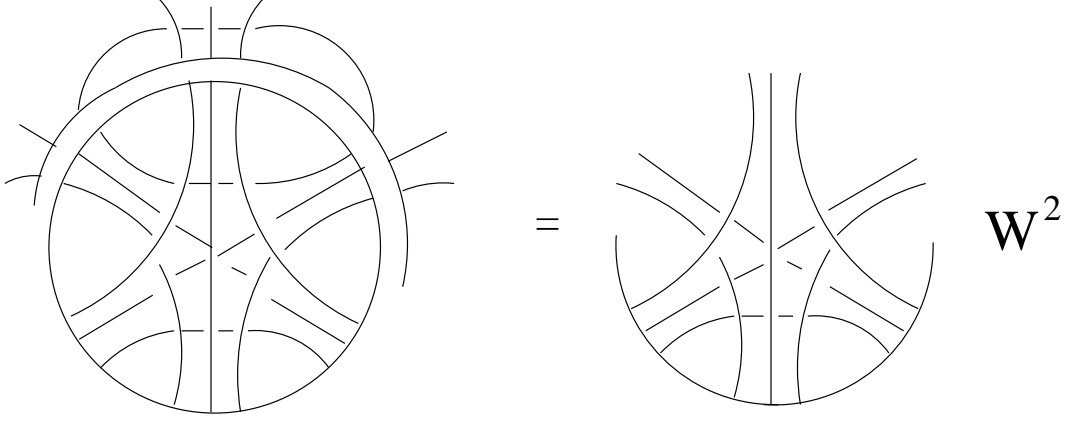


Figure 17: Shelling of $[3 \rightarrow 3]_{ish}^5$ type.

FIG.18, where we see three loops, the contributions of which cancel the above extra

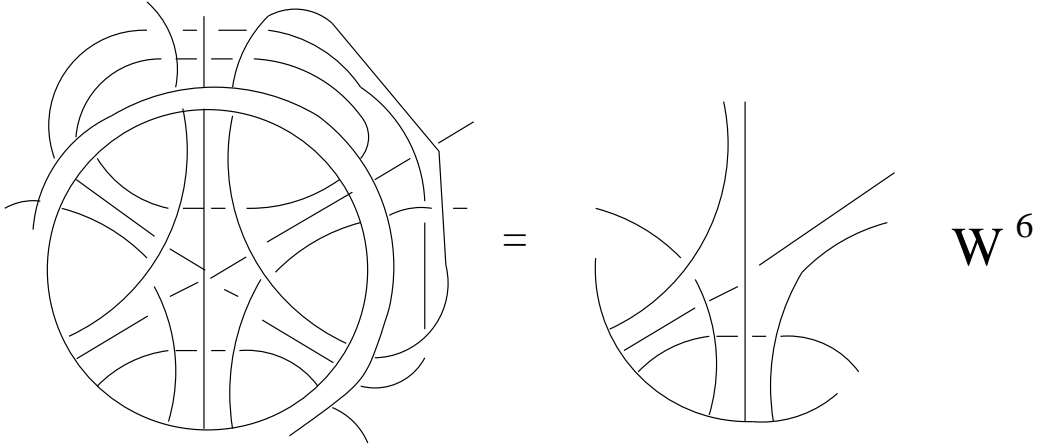


Figure 18: Shelling of $[2 \rightarrow 4]_{ish}^5$ type.

factor.

The last type of inverse shelling that we deal with is $[1 \rightarrow 5]_{ish}^5$, representing the glueing of a 5-simplex along one 4-simplex. By counting in an appropriate way the subsimplices of the new configuration in ∂T^5 , we see that there appear ten triangles $((d-3)$ -simplices), five edges $((d-4)$ -simplices) and one vertex $((d-5)$ -simplex), which generate an overall w_L^{-12} factor. Looking now at FIG.19, such extra factor turns out to be exactly cancelled by the contributions arising from the loops. This completes the proof of the invariance of (46) under elementary boundary operations in the 5-dimensional case.

Coming to the general d -dimensional case, we slightly change our previous notation, namely $[n_{d-1} \rightarrow d - (n_{d-1})]_{ish}^d$ ($n_{d-1} = 1, 2, \dots, d$), by parametrizing the moves in terms of d according to

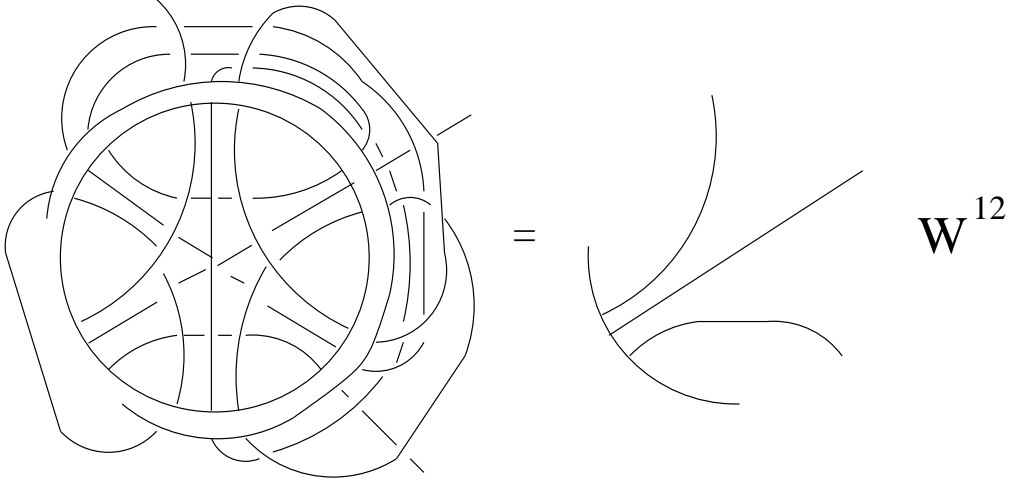


Figure 19: Shelling of $[1 \rightarrow 5]_{ish}^5$ type.

$$[(d-k) \rightarrow (k+1)]_{ish}^d, k = 0, 1, 2, \dots, (d-1), \quad (47)$$

which of course turns out to be consistent with the previous one. Notice also that in what follows we shall make use of the diagrammatic relations shown in FIG.9 and FIG.10 whenever it is necessary.

Consider first $[d \rightarrow 1]_{ish}^d$, representing the glueing of a d -simplex to ∂T^d along d $(d-1)$ -dimensional simplices (the corresponding diagram is given in FIG.20). Since

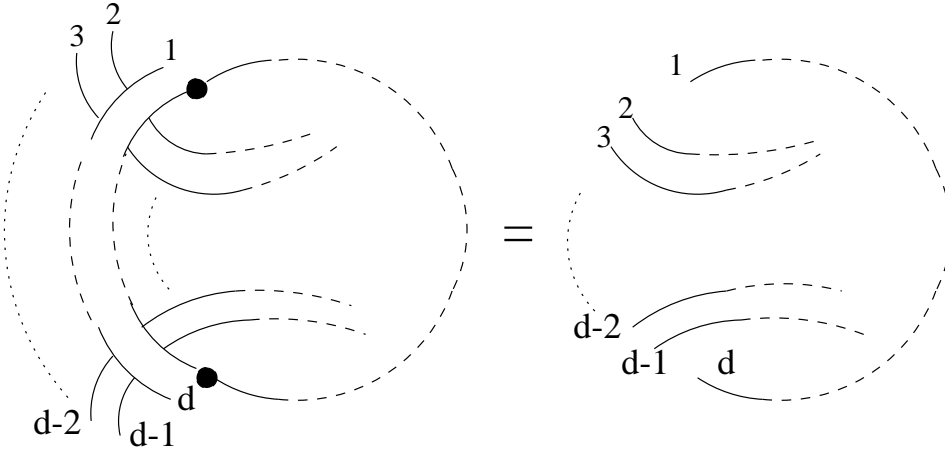


Figure 20: Shelling of $[d \rightarrow 1]_{ish}^d$ type.

we do not generate any $(d-r)$ -simplex ($r \geq 3$) in the new configuration $\partial T'^d$, no additional w_L^{-2} factors enter the state sum (45).

The $[(d-1) \rightarrow 2]_{ish}^d$ move consists in the attachment of a d -simplex to ∂T^d along $(d-1)$ simplices of dimension $(d-1)$. Also the action of this inverse shellings does

not give any $(d-r)$ -simplex ($r \geq 3$) in the new $\partial T'^d$, and its graphical counterpart is shown in FIG.21.

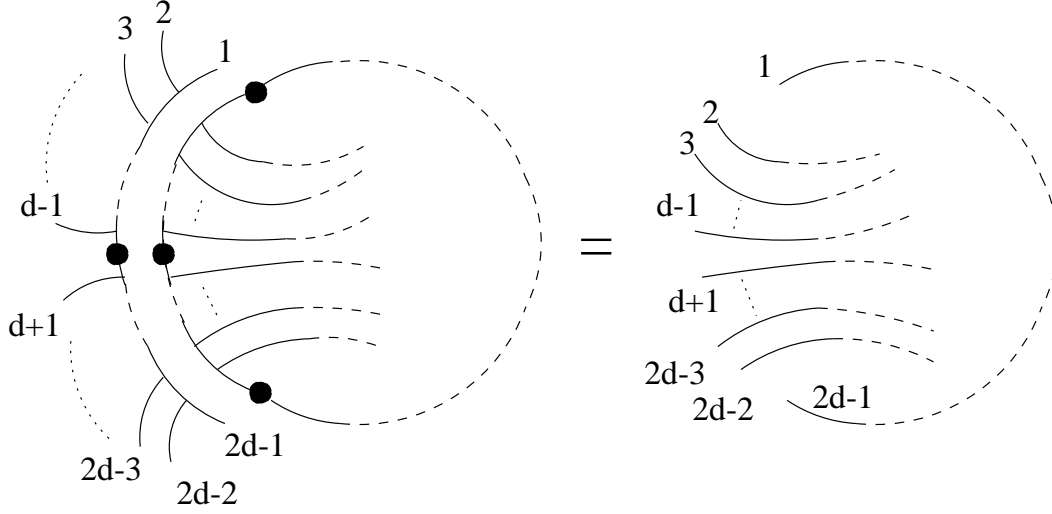


Figure 21: Shelling of $[d-1 \rightarrow 2]_{ish}^d$ type.

Coming to $[(d-2) \rightarrow 3]_{ish}^d$, we see that it represents the glueing of a d -simplex along $(d-2)$ simplices of dimension $(d-1)$ lying in ∂T^d and its diagram is given in FIG.22. In the new boundary triangulation one $(d-3)$ -simplex (and no other

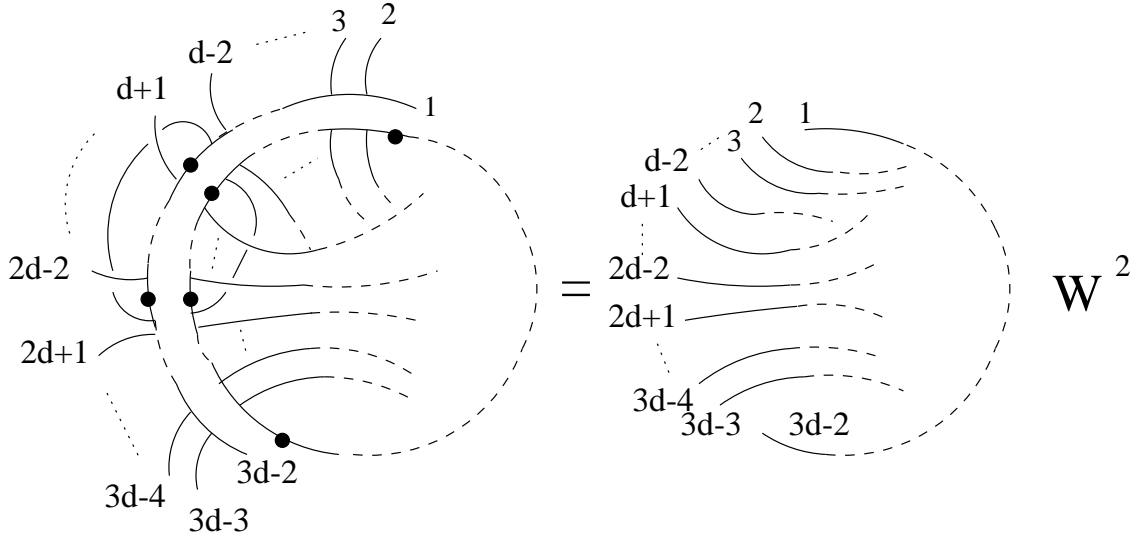


Figure 22: Shelling of $[d-2 \rightarrow 3]_{ish}^d$ type.

simplices) appears. Thus the w_L^{-2} factor which comes out is exactly cancelled by the contribution of the loop.

An algorithmic setting can now be easily established. For, the k -th inverse shelling, namely $[(d-k) \rightarrow (k+1)]_{ish}^d$, represents the attachment of a new d -simplex along

$(d - k)$ simplices of dimension $(d - 1)$ in ∂T^d which generates in $\partial T'^d$ several kinds of new boundary components. The number of such new components is evaluated by using suitable binomial coefficients, according to the list given below

$$\left(\begin{matrix} k+1 \\ 3 \end{matrix} \right) \sigma^{d-3}; \left(\begin{matrix} k+1 \\ 4 \end{matrix} \right) \sigma^{d-4}; \dots; \left(\begin{matrix} k+1 \\ k+1 \end{matrix} \right) \sigma^{d-(k+1)}, \quad (48)$$

while for each k the following number of additional w_L^{-2} factors are generated

$$w_L^{2 \sum_{i=3}^{k+1} \left(\begin{matrix} k+1 \\ i \end{matrix} \right) (-1)^{i+1}} = w_L^{-2 \frac{k-1}{2} k}. \quad (49)$$

The action of the k -th inverse shelling is depicted in the diagram of FIG.23, where

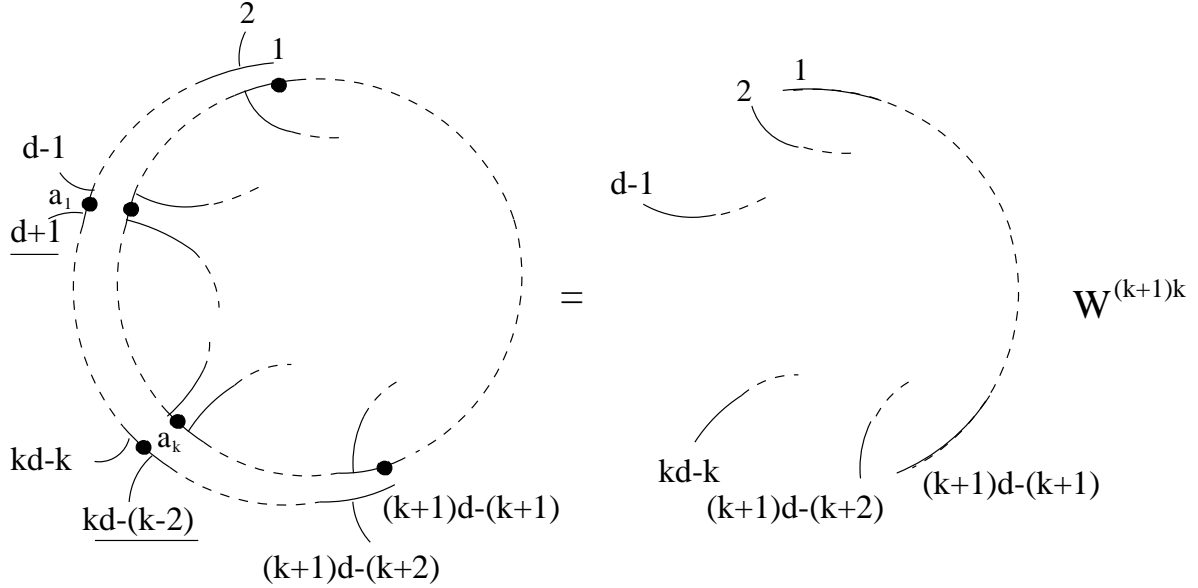


Figure 23: Shelling of $[d - k \rightarrow k + 1]_{ish}^d$ type.

the loops which cancel the weights given in (49) appear. More precisely, when we glue together $(k+1)$ $(d-1)$ -simplices in the more symmetric way, we have to perform $(\sum_{l=1}^k l)$ identifications among their faces (and that is exactly the number of summations over j variables in the state sum). However, k of the above identifications do not bring loops (as can be inferred from the structure of the $\{3(d-2)(d+1)/2\}j$ symbol) and thus the remaining $(\sum_{l=1}^{k-1} l)$ glueing operations induce exactly the factors given in (49). This remark completes the proof of the equivalence of (45) under the entire set of boundary elementary operations. Thus, by Pachner's result found in [10], the expression given in (46) is formally an invariant of the PL -structure of $(M^d, \partial M^d)$. Its regularized counterpart, $Z[(M^d, \partial M^d)](q)$ can be written explicitly according to the prescription given in *Appendix B*.

6 Invariants of closed M^d from colorings of $(d-2)$ -simplices

In order to complete the program outlined in the introduction, in the present section we show how it is possible to generate from the extended invariant defined in *Section 5* a state sum model for a closed PL -manifold M^d . Obviously the form of the new state sum will be inferred from the extended one (given in (45)) simply by ignoring the contributions from boundary (*cfr.* the 3 and 4-dimensional cases). Thus the main point of the present section will consist in giving the proof that (46), and consequently $Z[M^d] \doteq Z[(M^d, \partial M^d)]|_{\partial M^d=\emptyset}$, are invariant under d -dimensional bistellar moves performed either in the interior (bulk) of a triangulation $(T^d, \partial T^d)$ or in a closed T^d , respectively.

The expression of the state sum for a triangulation T^d , colored according to the prescription of the previous section and with the same notations, reads

$$\begin{aligned} Z[T^d(j_{\sigma^{d-2}}, J_{\sigma^{d-1}}) \rightarrow M^d; L] = \\ = w_L^{(-1)^{d\Xi}} \prod_{\text{all } \sigma^{d-2}} (-1)^{2j_{\sigma^{d-2}}} (2j_{\sigma^{d-2}} + 1) \prod_{\text{all } \sigma^{d-1}} \left(\prod_{C=1}^{d-3} (-1)^{2J_C} (2J_C + 1) \right)_{\sigma^{d-1}} \cdot \\ \cdot \prod_{\text{all } \sigma^d} \left\{ \frac{3}{2} (d-2)(d+1)j \right\}_{\sigma^d} (J), \end{aligned} \quad (50)$$

while the limit taken on all admissible colored triangulations of a given PL -manifold M^d is formally written as follows

$$Z[M^d] = \lim_{L \rightarrow \infty} \sum_{\left\{ \begin{array}{c} T^d \\ j, J \leq L \end{array} \right\}} Z[T^d(j, J) \rightarrow M^d; L]. \quad (51)$$

The issue of the equivalence of (51) under the suitable set of topological operations can be addressed by exploiting some results from PL -topology (recall also the definitions given at the beginning of the introduction) and on applying them to the extended state sum $Z^d[(T^d, \partial T^d) \rightarrow \dots]$ given in (45).

Let us start by considering the simplicial complex made up by one d -simplex σ^d together with all its subsimplices. From the topological point of view we get in fact what is called a (standard) simplicial PL -ball, and we denote it by $\mathcal{B}^d(\sigma)$ (we omit the dimensionality of simplices whenever it is clear from the context). The boundary of such a ball, $\partial \mathcal{B}^d(\sigma)$, is obviously homeomorphic to the $(d-1)$ -dimensional sphere and in particular it is a simplicial PL -sphere containing the $(d+1)$ faces of dimension $(d-1)$ of the original $\mathcal{B}^d(\sigma)$. Notice however that a simplicial $(d-1)$ -sphere can be defined by its own by joining in a suitable way some $(d-1)$ -dimensional simplices $\subset \mathbb{R}^d$. The minimum number of $(d-1)$ -simplices necessary to get a PL -sphere is just $(d+1)$: the resulting simplicial sphere will be denoted by $\mathcal{S}^{d-1}(\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_{d+1})$.

If we consider again the PL -ball $\mathcal{B}^d(\sigma)$, we would get $\partial\mathcal{B}^d(\sigma) \cong_{PL} \mathcal{S}^{d-1}(\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_{d+1})$, where \cong_{PL} stands for a PL -homeomorphism.

Turning now to the structure of the extended state sum for a $(T^d, \partial T^d)$, we can reconsider its topological content as follows. Indeed, we see that the contribution of the configuration $\mathcal{S}^{d-1}(\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_{d+1})$ to Z^d amounts exactly to one $\{3(d-2)(d+1)/2\}j$ symbol, namely it is the same that we would obtain by glueing a $\mathcal{B}^d(\sigma)$ to $\mathcal{S}^{d-1}(\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_{d+1})$ along $\partial\mathcal{B}^d(\sigma)$ with a PL -homeomorphism. The reason why we stress this point relies on the fact that on this basis we are able to set up the following step-by-step procedure: *i*) we extract first some $\mathcal{B}^d(\sigma)$ from the bulk of $(T^d, \partial T^d)$, leaving an internal hole bounded by the PL -sphere $\mathcal{S}^{d-1}(\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_{d+1})$; *ii*) then we carry out elementary boundary operations on the PL -pair $(\mathcal{B}^d(\sigma), \partial\mathcal{B}^d(\sigma))$ bringing $\partial\mathcal{B}^d(\sigma)$ into $\mathcal{S}^{d-1}(\tau_1 \cup \tau_2 \cup \dots \cup \tau_{d+1})$ (notice that in doing that we do not alter the extended state sum, owing to its invariance under elementary shellings); *iii*) finally, we glue the ball back into the original triangulation through a PL -homeomorphism $\mathcal{S}^{d-1}(\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_{d+1}) \cong_{PL} \mathcal{S}^{d-1}(\tau_1 \cup \tau_2 \cup \dots \cup \tau_{d+1})$.

Such kinds of *cut and paste* represent nothing that the implementation of the set of d -dimensional bistellar moves in the bulk of each triangulation of $(M^d, \partial M^d)$ (and in the whole closed M^d). To be precise, the entire set of moves will be obtained by cutting away not just a standard PL -ball as before, but rather simplicial balls made up of a suitable collection of more than one d -simplex.

As an explicit example of how the above procedure works, consider the 3-dimensional case with the corresponding extended state sum given in (7). Recall from (10) that in this case we are dealing with four bistellar moves, $[1 \leftrightarrow 4]_{bst}^3$ and $[2 \leftrightarrow 3]_{bst}^3$, where the arguments refer to the number of 3-simplices involved in the corresponding transformation. The explicit implementation of some of these moves is given below.

- $[1 \rightarrow 4]_{bst}^3$. Since the initial configuration contains just one 3-simplex, we are just in the situation described above. Then we extract the ball $\mathcal{B}^3(\sigma^3)$ (the boundary of which is $\mathcal{S}^2(\sigma_1^2 \cup \sigma_2^2 \cup \sigma_3^2 \cup \sigma_4^2)$) and perform on it the inverse shelling $[1 \rightarrow 3]_{ish}^3$, where the first triangle is chosen in an arbitrary way. Thus we get a configuration with two 3-simplices, namely the original σ^3 and a new τ_1^3 , glued along the original triangle. The second operation is an inverse shelling $[2 \rightarrow 2]_{ish}^3$, where the two initial contiguous triangles belong to σ^3 and to τ_1^3 , respectively. This move generates a third tetrahedron, τ_2^3 , joined to the previous ones through a 2-dimensional face. On this configuration we act now with $[3 \rightarrow 1]_{ish}^3$, where two of the three triangles of the initial arrangement were generated in the two previous steps, respectively, while the third one belongs to the original σ^3 . Thus we get a fourth tetrahedron τ_3^3 which, together with the other ones, gives rise to the simplicial ball $\mathcal{B}^3(\sigma^3 \cup \tau_1^3 \cup \tau_2^3 \cup \tau_3^3)$, the boundary of which is $\mathcal{S}^2(\sigma^2 \cup \tau_{(1)}^2 \cup \tau_{(2)}^2 \cup \tau_{(3)}^2)$, where the first entry is the initial triangle chosen in σ^3 , and the other entries are the new faces generated in the three previous steps ($\tau_{(1)}^2 \subset \tau_1^3; \dots$). Now we glue back the modified simplicial ball

into the triangulation through a PL -homeomorphism between the original \mathcal{S}^2 ($\sigma_1^2 \cup \sigma_2^2 \cup \sigma_3^2 \cup \sigma_4^2$) and $\mathcal{S}^2(\sigma^2 \cup \tau_{(1)}^2 \cup \tau_{(2)}^2 \cup \tau_{(3)}^2)$. The pictorial representation of the reconstruction of this particular move is shown in FIG.24.

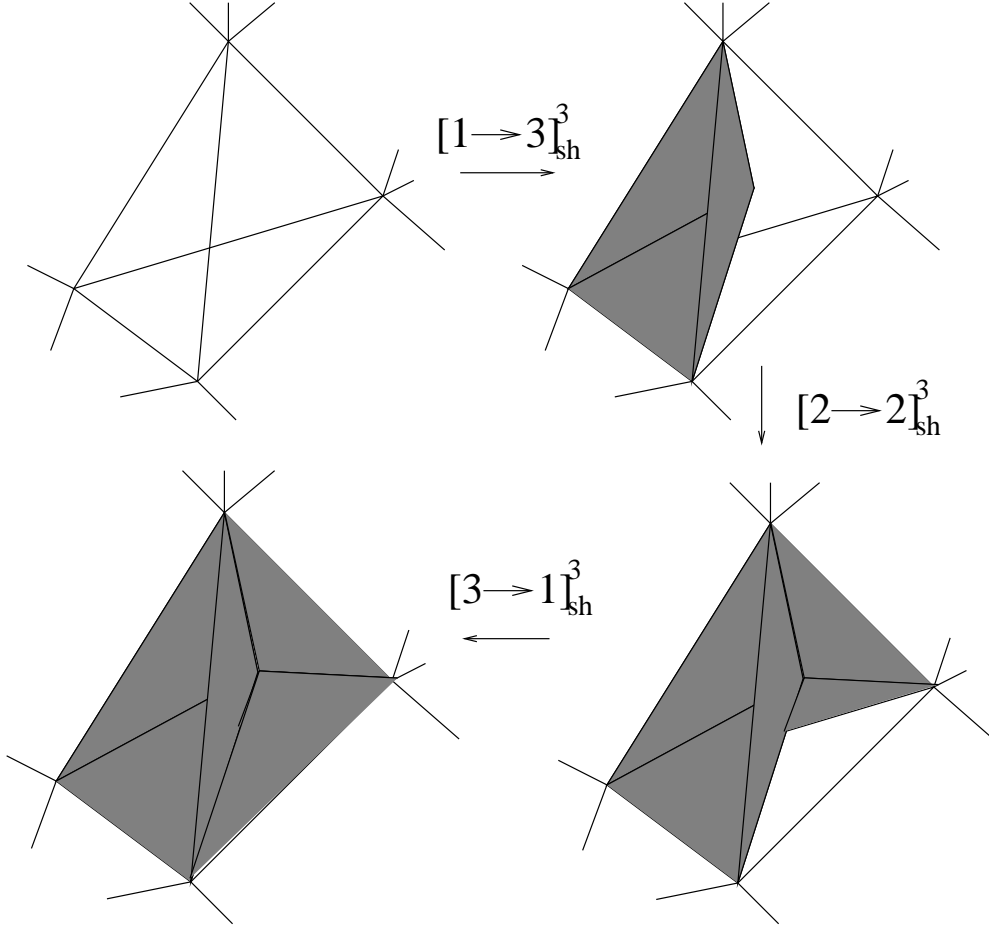


Figure 24: The move $[1 \rightarrow 4]_{bst}^3$ in terms of inverse shellings performed on the removed PL -ball.

- $[2 \rightarrow 3]_{bst}^3$. The configuration we start with is a ball $\mathcal{B}^3(\sigma_1^3 \cup \sigma_2^3)$ having $\mathcal{S}^2(\sigma_1^2 \cup \sigma_2^2 \cup \sigma_3^2 \cup \sigma_4^2 \cup \sigma_5^2 \cup \sigma_6^2)$ as its boundary. After the extraction, we perform first the inverse shelling $[2 \rightarrow 2]_{ish}^3$, where the two initial triangles are contiguous and belong to σ_1^3 and to σ_2^3 , respectively. Finally, we apply $[3 \rightarrow 1]_{ish}^3$, where the three initial triangles belong to σ_1^3 , σ_2^3 and to the component generated by the previous step, respectively. It is not difficult to realize that the resulting PL -ball has again six faces in its boundary PL -sphere, and thus we glue it back along the boundary of the original hole by means of a suitable PL -homeomorphism. The sequence of operations we have performed is drawn in FIG.25.

The remaining 3-dimensional bistellar moves can be explicitly worked out following

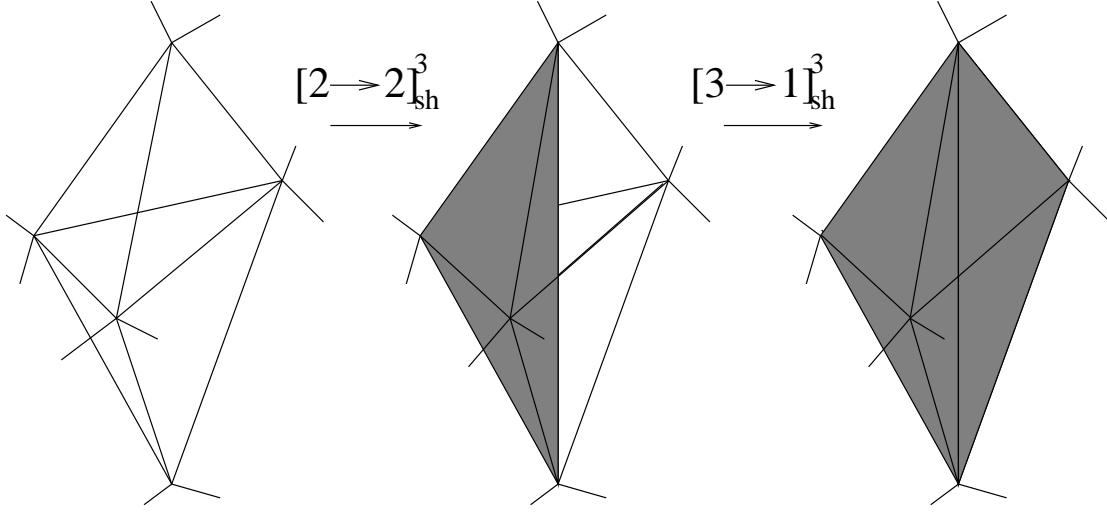


Figure 25: The move $[2 \rightarrow 3]_{bst}^3$ in terms of inverse shellings performed on the removed PL-ball.

a similar procedure and by employing a definite sequence of inverse operations. Also for what concerns the 4-dimensional case we could describe step-by-step the implementation of all transformation. However, since in that case a pictorial counterpart is not so easily displayed, we turn to the general rule in dimension d . On the basis of the characterization of d -dimensional inverse shellings given in (47) we can reparametrize also the set of all d -bistellar moves according to

$$[k \rightarrow (d - k + 1)]_{bst}^d ; k = 1, 2, \dots, (d - 1), \quad (52)$$

paying attention to the fact that here k enumerates d -simplices, while in (47) it was referred to $(d - 1)$ -simplices in a given σ^d . With this reformulation, we see that the generic move $[k \rightarrow (d - k + 1)]_{bst}^d$ can be obtained with the *cut and paste* procedure following the steps:

- remove from the bulk of $(T^d, \partial T^d)$ a PL -ball $\mathcal{B}^d(\sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_k)$, where $k = 1, 2, \dots, (d - 1)$;
- implement on the ball the sequence of inverse shellings $[k \rightarrow d - k + 1]_{ish}^d, [k + 1 \rightarrow d - k]_{ish}^d, \dots, [d + 1 \rightarrow 1]_{ish}^d$, by choosing in the initial configuration one $(d - 1)$ -simplex in each of the components of the original ball and by involving in the subsequent moves one $(d - 1)$ -simplex for each of the components generated before (including also the initial one);
- glue back the resulting modified simplicial ball along the hole left in $(T^d, \partial T^d)$ by using a PL -homeomorphism.

Having shown that the extended state sum is equivalent under (a finite set of) bistellar moves performed in the bulk for any dimension d , we can conclude that

the expression of $Z[M^d]$ in (51) share the same property. Thus, owing to [9], it is also an invariant of the PL -structure. For what concern the q -deformed $Z[M^d](q)$, we refer as usual to the notations and definitions collected in *Appendix B*.

Conclusions

Coming back to the array of PL -invariants displayed in the introduction, we would hopefully complete it with a third hierarchy, namely

$$\begin{array}{cccc}
 \textit{dimension} : & 2 & 3 & 4 & \dots \\
 & Z_\chi^2 & Z_{PR}^3 & W^4 & \\
 & & Z_\chi^3 & Z_{CY}^4 & \dots \\
 & & & & \dots
 \end{array}$$

As we learnt from the results found in the previous sections, the key tool to build up the first entry of the new hierarchy, namely W^4 , consists in rewriting Z_{PR}^3 in terms of double symbols (to be associated with each tetrahedron). We are currently investigating such a possibility, which could become concrete by exploiting the Regge symmetries of the $6j$ symbol (see [13]). The next step will consist in picking up one of the sub-symbols of the double symbol, collecting five of them in a suitable way, and associating the resulting expression with the fundamental 4-dimensional block. Finally, the issue of the equivalence of the resulting state sum under moves should be improved, together with the identification of the *continuous* counterpart of the new PL -invariant. Indeed, such a W^4 could be non-trivial since it would be easily extended to an invariant for a pair $(M^4, \partial M^4)$ having Z_{PR}^3 on its boundary manifold.

As a last remark, we would like to spend some words on the procedure described in *Section 6*, where we established that the equivalence of the closed state sum under bistellar moves follows from the equivalence of the extended state sum under shellings. It is worthwhile to notice that the possibility of carrying out topological operations on removed PL -balls relies on a theorem given in [11] which states that *Every d -dimensional PL -ball is shellable*. On the other hand, it is clear that the glueings of the modified balls into the bulk of the original triangulation are achieved through PL -homeomorphisms which are consistent with respect to $SU(2)$ -labellings but, generally speaking, are not at all isometric mappings (the natural metric being an Euclidean PL -metric induced on the underlying polyhedron). Thus, if we were interested in finding a state sum for a pair $(M^d, \partial M^d)$ in which the edge lengths of the simplices in the bulk are dynamical variables (whereas for instance we would keep on requiring invariance on the boundary manifold) then we should accordingly modify the whole approach. We are currently addressing such issues in connection

with the search for discretized models of Euclidean quantum gravity in dimension four.

Appendix A

We give here explicitly the identities corresponding to the four types of *elementary shellings* in $d = 4$ which are employed to show the invariance of $Z[(M^4, \partial M^4)]$ in *Section 3*. Recall from (29) and (31) that there we made use of primed spin variables $\{j', J'\}$ in order to label those components which lie in ∂T^4 in some configuration, while plain $\{j, J\}$ denoted components in $\text{int}(T^4)$. Since here almost all variables are indeed in ∂T^4 , we agree to change our previous notation according to

$$\begin{aligned} j'_1, j'_2, \dots, j'_{10} &\longrightarrow j_1, j_2, \dots, j_{10}, \\ J'_a(\subset \partial T^4) &\longrightarrow J_a, \\ J_a(\subset \text{int}(T^4)) &\longrightarrow \mathbf{J}_a. \end{aligned}$$

For the corresponding m variables we keep on denoting them by plain m_1, m_2, \dots, m_{10} ; m_a, \dots, m_e , since no ambiguity can arise. We set also $w_j^2 \equiv (2j + 1)$, and the summation labels and arguments are shortened as far as possible.

$[1 \rightarrow 4]_{\text{sh}}^4$ (see FIG.5)

$$\begin{aligned} \sum_{J_a, m_a} w_{J_a}^2 (-1)^{m_a} \begin{pmatrix} j_1 & j_2 & J_a \\ m_1 & m_2 & m_a \end{pmatrix} \begin{pmatrix} j_3 & j_4 & J_a \\ m_3 & m_4 & -m_a \end{pmatrix} [J_a, \mathbf{J}_b, \mathbf{J}_c, \mathbf{J}_d, \mathbf{J}_e] = \\ \cdot \sum_{m_i} (-1)^{\sum_i m_i} \sum_{m_A} (-1)^{\sum_A m_A} \begin{pmatrix} j_5 & j_6 & J_b \\ m_5 & m_6 & m_b \end{pmatrix} \begin{pmatrix} j_3 & j_7 & J_b \\ -m_3 & m_7 & -m_b \end{pmatrix} \cdot \\ \cdot \begin{pmatrix} j_5 & j_8 & J_c \\ -m_5 & m_8 & m_c \end{pmatrix} \begin{pmatrix} j_1 & j_9 & J_c \\ -m_1 & m_9 & -m_c \end{pmatrix} \begin{pmatrix} j_6 & j_{10} & J_d \\ -m_6 & m_{10} & m_d \end{pmatrix} \cdot \\ \cdot \begin{pmatrix} j_2 & j_8 & J_d \\ -m_2 & -m_8 & -m_d \end{pmatrix} \begin{pmatrix} j_7 & j_{10} & J_e \\ -m_7 & -m_{10} & m_e \end{pmatrix} \begin{pmatrix} j_4 & j_9 & J_e \\ -m_4 & -m_9 & -m_e \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \{m_i\} &= (m_5, m_6, m_7, m_8, m_9, m_{10}) \\ \{m_A\} &= (m_b, m_c, m_d, m_e). \end{aligned}$$

$[2 \rightarrow 3]_{\text{sh}}^4$ (see FIG.6)

$$\begin{aligned}
& \sum_{j_3, m_3} w_{j_3}^2 (-1)^{m_3} \sum_{J_A, m_A} \left(\prod_A w_{J_A}^2 \right) (-1)^{\sum_A m_A} \begin{pmatrix} j_1 & j_2 & J_a \\ m_1 & m_2 & m_a \end{pmatrix} \begin{pmatrix} j_3 & j_4 & J_a \\ m_3 & m_4 & -m_a \end{pmatrix} \\
& \cdot \begin{pmatrix} j_5 & j_6 & J_b \\ m_5 & m_6 & m_b \end{pmatrix} \begin{pmatrix} j_3 & j_7 & J_b \\ -m_3 & m_7 & -m_b \end{pmatrix} [J_a, J_b, \mathbf{J}_c, \mathbf{J}_d, \mathbf{J}_e] = \\
& = \sum_{m_i} (-1)^{\sum_i m_i} \sum_{m_A} (-1)^{\sum_A m_A} \begin{pmatrix} j_5 & j_8 & J_c \\ -m_5 & m_8 & m_c \end{pmatrix} \begin{pmatrix} j_1 & j_9 & J_c \\ -m_1 & m_9 & -m_c \end{pmatrix} \\
& \cdot \begin{pmatrix} j_6 & j_{10} & J_d \\ -m_6 & m_{10} & m_d \end{pmatrix} \begin{pmatrix} j_2 & j_8 & J_d \\ -m_2 & -m_8 & -m_d \end{pmatrix} \begin{pmatrix} j_7 & j_{10} & J_e \\ -m_7 & -m_{10} & m_e \end{pmatrix} \\
& \cdot \begin{pmatrix} j_4 & j_9 & J_e \\ -m_4 & -m_9 & -m_e \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\{J_A\} &= (J_a, J_b) \\
\{m_A\} &= (m_a, m_b) \\
\{m_i\} &= (m_8, m_9, m_{10}) \\
\{m_B\} &= (m_c, m_d, m_e).
\end{aligned}$$

$[3 \rightarrow 2]_{\text{sh}}^4$ (see FIG.7)

$$\begin{aligned}
& \sum_{j_i, m_i} \left(\prod_i w_{j_i}^2 \right) (-1)^{\sum_i m_i} \sum_{J_A, m_A} \left(\prod_A w_{J_A}^2 \right) (-1)^{\sum_A m_A} \begin{pmatrix} j_1 & j_2 & J_a \\ m_1 & m_2 & m_a \end{pmatrix} \\
& \cdot \begin{pmatrix} j_3 & j_4 & J_a \\ m_3 & m_4 & -m_a \end{pmatrix} \begin{pmatrix} j_5 & j_6 & J_b \\ m_5 & m_6 & m_b \end{pmatrix} \begin{pmatrix} j_3 & j_7 & J_b \\ -m_3 & -m_7 & -m_b \end{pmatrix} \\
& \cdot \begin{pmatrix} j_5 & j_8 & J_c \\ -m_5 & m_8 & m_c \end{pmatrix} \begin{pmatrix} j_1 & j_9 & J_c \\ -m_1 & m_9 & -m_c \end{pmatrix} [J_a, J_b, J_c, \mathbf{J}_d, \mathbf{J}_e] = \\
& = w_L^2 \sum_{m_{10}} (-1)^{m_{10}} \sum_{m_B} (-1)^{\sum_B m_B} \begin{pmatrix} j_6 & j_{10} & J_d \\ -m_6 & m_{10} & m_d \end{pmatrix} \begin{pmatrix} j_2 & j_8 & J_d \\ -m_2 & -m_8 & -m_d \end{pmatrix} \\
& \cdot \begin{pmatrix} j_7 & j_{10} & J_e \\ -m_7 & -m_{10} & m_e \end{pmatrix} \begin{pmatrix} j_4 & j_9 & J_e \\ -m_4 & -m_9 & -m_e \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\{j_i\} &= (j_1, j_3, j_5) \\
\{m_i\} &= (m_1, m_2, m_5) \\
\{J_A\} &= (J_a, J_b, J_c) \\
\{m_A\} &= (m_a, m_b, m_c) \\
\{m_B\} &= (m_d, m_e) \\
w_L^2 &= \Lambda(L).
\end{aligned}$$

$[4 \rightarrow 1]_{\text{sh}}^4$ (see FIG.8)

$$\begin{aligned}
& \sum_{j_i, m_i} \left(\prod_i w_{j_i}^2 \right) (-1)^{\sum_i m_i} \sum_{J_A, m_A} \left(\prod_A w_{J_A}^2 \right) (-1)^{\sum_A m_A} \begin{pmatrix} j_1 & j_2 & J_a \\ m_1 & m_2 & m_a \end{pmatrix} \\
& \cdot \begin{pmatrix} j_3 & j_4 & J_a \\ m_3 & m_4 & -m_a \end{pmatrix} \begin{pmatrix} j_5 & j_6 & J_b \\ m_5 & m_6 & m_b \end{pmatrix} \begin{pmatrix} j_3 & j_7 & J_b \\ -m_3 & m_7 & -m_b \end{pmatrix} \\
& \cdot \begin{pmatrix} j_5 & j_8 & J_c \\ -m_5 & m_8 & m_c \end{pmatrix} \begin{pmatrix} j_1 & j_9 & J_c \\ -m_1 & m_9 & -m_c \end{pmatrix} \begin{pmatrix} j_6 & j_{10} & J_d \\ -m_6 & m_{10} & m_d \end{pmatrix} \\
& \cdot \begin{pmatrix} j_2 & j_8 & J_d \\ -m_2 & -m_8 & -m_d \end{pmatrix} [J_a, J_b, J_c, J_d, \mathbf{J}_e] = \\
& = w_L^6 \sum_{m_e} (-1)^{m_e} \begin{pmatrix} j_7 & j_{10} & J_e \\ -m_7 & -m_{10} & m_e \end{pmatrix} \begin{pmatrix} j_4 & j_9 & J_e \\ -m_4 & -m_9 & -m_e \end{pmatrix} \\
& \{j_i\} = (j_1, j_2, j_3, j_5, j_6, j_8) \\
& \{m_i\} = (m_1, m_2, m_3, m_5, m_6, m_8) \\
& \{J_A\} = (J_a, J_b, J_c, J_d) \\
& \{m_A\} = (m_a, m_b, m_c, m_d) \\
& w_L^6 = \Lambda(L)^3.
\end{aligned}$$

The full set of identities can be obtained, up to regularization, from anyone of them, on applying orthogonality/completeness conditions for $3jm$ symbols as well as orthogonality conditions for the $15j$ symbol (see [21]). Moreover, the set of inverse moves (corresponding to the attachment of a 4-simplex) can be read in the same set of identities up to exchanging the role of J and \mathbf{J} .

Appendix B

All state sums and associated classical invariants defined in terms of (re)coupling coefficients of $SU(2)$ can be extended to the case of the quantum enveloping algebra $U_q(sl(2, \mathbb{C}))$, q a root of unity. Following the standard notation, the spin variables $\{j\}$ take their values in a finite set $I = (0, 1/2, 1, \dots, k)$ where $\exp(\pi i/k) = q$. For each $j \in I$ a function $w_{(q)j}^2 \doteq (-1)^{2x_j} [2x_j + 1]_q \in K^*$ is defined, where $K^* \equiv K \setminus \{0\}$ (K a commutative ring with unity). Recall that the notation $[\cdot]_q$ stands for a q -integer, namely $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ and that, for each admissible triple (j, k, l) , we have: $w_{(q)j}^{-2} \sum_{k,l} w_{(q)k}^2 w_{(q)l}^2 = w_q^2$, with $w_q^2 = -2k/(q - q^{-1})^2$.

We do not give the explicit expression of the quantum invariants $Z[(M^d, \partial M^d)](q)$ and $Z[M^d](q)$ which would replace the classical counterparts found in *Section 5* and *Section 6*. We just notice that the basic receipt to transform the classical state sums into the quantum ones can be summarized as follows

- the classical weights $(-1)^{2j}(2j+1)$ are replaced by $w_{(q)j}^2$, while each of the factors $\Lambda(L)^{-1}$ becomes w_q^{-2} ;
- each Wigner symbol of $SU(2)$ is replaced by its q -analog $q-3jm$, normalized as explained below;
- each classical recoupling coefficient (or $3nj$ symbol) of a given type has its q -deformed counterpart, obtained by summing over magnetic numbers products of $q-3jm$ symbols, apart from suitable phase factors.

Recall from [6] and [7] that the relation between the quantum Clebsh–Gordan coefficient $(j_1 m_1 j_2 m_2 | j_3 m_3)_q$ and the $q-3jm$ symbol is given by

$$(j_1 m_1 j_2 m_2 | j_3 m_3)_q = (-1)^{j_1 - j_2 + m_3} ([2j_3 + 1]_q)^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}_q, \quad (53)$$

where, as usual, an m variable runs in integer steps between $-j$ and $+j$, and the classical expression is recovered when $q = 1$. The symmetry properties of the $q-3jm$ symbol read

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}_q &= (-1)^{j_1 + j_2 + j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & -m_3 \end{pmatrix}_{1/q}, \\ \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}_q &= (-1)^{j_1 + j_2 + j_3} q^{-m_1/2} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & -m_2 \end{pmatrix}_{1/q}, \\ \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}_q &= (-1)^{j_1 + j_2 + j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & m_3 \end{pmatrix}_q. \end{aligned} \quad (54)$$

Thus we define the *normalized* $q-3jm$ symbols, for deformation parameters q and $1/q$ respectively, according to

$$\begin{aligned} \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{bmatrix}_q &\doteq q^{(m_1 - m_2)/6} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}_q \\ \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{bmatrix}_{1/q} &\doteq q^{(m_2 - m_1)/6} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}_{1/q} \end{aligned} \quad (55)$$

The orthogonality relation involving the normalized symbols (which are used for instance in order to handle identities representing elementary shellings and inverse shellings) reads

$$\sum_{jm} w_{(q)j}^2 (-1)^\theta q^{(m_2 - m_1)/3} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{bmatrix}_q \begin{bmatrix} j_2 & j_1 & j \\ -m'_2 & -m'_1 & -m \end{bmatrix}_q = \delta_{m_1 m'_1} \delta_{m_2 m'_2}, \quad (56)$$

where $\theta = m_1 + m_2 + m_3$.

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